MAT 536, Spring 2024
PROBLEM SET 7, Due Wednesday, April 3

Problems are due in class on Mondays (if you can't attend class, email solutions to the grader before class begins). Please restate problem at each beginning of your solution, or attach this sheet to the front of your solution set. Solutions should be legible and written in complete, correct sentences. Handwritten or a PDF from LaTeX is acceptable. Cite any sources that you use, e.g., a Wikipedia article or another textbook (most problems will not require outside sources).

(1) Suppose $\{f_n\}$ are analytic maps of a connected, open set Ω into $\mathbb{D} = \{|z| < 1\}$. Suppose $\{f_n\}$ converges at infinitely many points of a compact set $K \subset \Omega$. Does $\{f_n\}$ necessarily converge at every point of Ω? Prove or give a counterexample.

(2) Let F_M be the collection of analytic on the unit disk that extend continuously to the unit circle and satisfy

$$\int_0^{2\pi} |f(e^{it}))|dt \leq M.$$

Prove F_M is a normal family with respect to the Euclidean metric.

(3) If f is 1-to-1 and holomorphic on the unit disk $\mathbb{D} = \{|z| < 1\}$, with $f(0) = 0$ and $f'(0) = 1$, use Schwarz's lemma to show that $f(\mathbb{D})$ does not contain all the unit circle $\mathbb{T} = \{|z| = 1\}$.

(4) Use the previous exercise to show that the collection of 1-to-1 analytic functions f on $\mathbb{D}$ so that $f(0) = 0$ and $f'(0) = 1$ is a normal family. (Hint: first consider the sub-collection of functions that never take the value 1 and apply Montel's theorem on $\mathbb{D} \setminus \{0\}$.)

(5) The Fatou set of a polynomial p is union of all open disks where the iterates form a normal family of meromorphic functions (the constant function ∞ is allowed as a limit). The Julia set is the complement of this. Show that the Julia set of a polynomial with degree ≥ 2 is non-empty.

(Hint: First prove there is an R so that every point z with $|z| > R$ iterates to infinity. If the Fatou set is the whole plane, and there is a sequence of iterates that converges to some function f uniformly on compact subsets of the plane. Derive a contradiction.)