MAT 536, Spring 2024
PROBLEM SET 6, Due Monday, March 4

Problems are due in class on Mondays (if you can't attend class, email solutions to the grader before class begins). Please restate problem at each beginning of your solution, or attach this sheet to the front of your solution set. Solutions should be legible and written in complete, correct sentences. Handwritten or a PDF from LaTeX is acceptable. Cite any sources that you use, e.g., a Wikipedia article or another textbook (most problems will not require outside sources).

(1) Give an explicit formula for the 1-1, onto analytic map from $\mathbb{D}$ to $\mathbb{D} \cap \mathbb{H}$ that fixes each of the points $1, -1, i$. ($\mathbb{D}$ is the unit disk, and $\mathbb{H}$ is the upper half-plane.)

(2) For $I = [a, b] \subset \mathbb{R}$ and $z \in \mathbb{H} = \{x + iy : y > 0\}$, let $\omega(z)$ be the angle subtended by the interval I from the point z. Show that ω harmonic in $\mathbb{H}$, tends to 0 at boundary points outside of I and tends to π at boundary points interior to I. This is the "harmonic measure" of I in the upper half-plane.

(3) Deduce the complex form of Green's theorem

$$\int_{\partial\Omega} f(z)dz = i \iint_\Omega (f_x + if_y)dxdy = 2i \iint_\Omega f_{\overline{z}}\, dxdy = \iint_\Omega f_{\overline{z}} d\overline{z} \wedge dz.$$

from the usual version: $\int_{\partial\Omega}(udx + vdy) = \iint_\Omega (v_y - u_x)dxdy$. (You may assume Ω and f are smooth $= C^\infty$).

(4) (Inhomogeneous Cauchy-Riemann equation) Suppose f is a smooth, compactly supported function on the plane. Use Green's theorem on an annulus $\{w : \epsilon < |w - z| < R\}$ to show

$$F(z) = \frac{1}{2\pi i} \iint \frac{f(w)}{w - z} dw \wedge d\overline{w} = -\frac{1}{\pi} \iint \frac{f(w)}{z - w} dxdy$$

is a smooth function so that $F_{\overline{z}} = f$. (Sometimes $F_{\overline{z}}$ is written $\overline{\partial} F$, and so $\overline{\partial} F = f$ is called a "D-bar equation". If F is a solution on a domain Ω, so if $F + H$ for any holomorphic H on Ω. Finding "good" solutions is often an important problem.)

(5) (Denjoy-Wolff Theorem) Suppose $f : \mathbb{D} \to \mathbb{D}$ is analytic, but not a linear fractional transformation. Then there is a $\alpha \in \overline{\mathbb{D}}$ so that $z_n = f^n(z) = f(f(\dots f(z)\dots))$ (n times) converges to α for every $z \in \mathbb{D}$. (Note that here f^n is the nth iterate of f, not its nth power.)

Hint: first assume there is a $r < 1$ so that $|z_n| \le r$ infinitely often and use fact that f strictly decreases the hypebolic metric (since it is not a LFT). If $|z_n| \to 1$, consider the map $f_\epsilon(z) = (1 - \epsilon)f(z)$. Show this has a fixed point z_ϵ, and $|z_\epsilon| \to 1$ as $\epsilon \to 0$. Let D_ϵ be the hyperbolic disk centered at z_ϵ with hyperbolic radius $\rho(0, z_\epsilon)$. This is also a Euclidean disk (but the Euclidean center is different than the hyperbolic center). If D is any limit if these disks as $\epsilon \searrow 0$ show that $f(D) \subset D$ and $\{z_n\}$ must converge to $\partial D \cap \partial\mathbb{D}$, which is a single point. This result is due to Denjoy and Wolff in separate 1926 papers, but this short proof is due to Beardon in 1990.