MAT 536, Spring 2024
PROBLEM SET 5, Due Monday, February 26

Problems are due in class on Mondays (if you can't attend class, email solutions to the grader before class begins). Please restate problem at each beginning of your solution, or attach this sheet to the front of your solution set. Solutions should be legible and written in complete, correct sentences. Handwritten or a PDF from LaTeX is acceptable. Cite any sources that you use, e.g., a Wikipedia article or another textbook (most problems will not require outside sources).

(1) Suppose a closed curve $\gamma : [0, 1] \to \mathbb{C} \setminus \{0\}$ has winding number n around the origin. Prove that γ hits each infinite ray R_θ starting at the origin at least n times (i.e., there are at least n distinct values so that $\arg(\gamma(t)) = \theta$).

(2) Suppose $a_0 > a_1 > \cdots > a_n > 0$. Show $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ has no zeros in $\overline{\mathbb{D}} = \{z : |z| \leq 1\}$. If $0 < a_0 < a_1 < \cdots < a_n$, show p has all its zeroes in $\mathbb{D}$. Hint: verify and apply the following facts to the terms of $(1 - z)p(z)$.
  (a) If $z \neq 0$ and $\arg(z) = \arg(z^2)$, then $z > 0$.
  (b) $|z_1 + \cdots + z_n| < |z_1| + \cdots + |z_n|$ unless $\arg z_1 = \cdots = \arg z_n$.

(3) Use the previous exercises to deduce that if $0 < a_0 < a_1 < \cdots < a_n$, then

$$f(z) = a_0 + a_1 \cos\theta + \cdots + a_n \cos n\theta$$

has at least $2n$ distinct zeros in $[0, 2\pi)$.

(4) Bloch functions are analytic functions on $\mathbb{D}$ so that $\sup_{z\in\mathbb{D}} |f'(z)|(1 - |z|) < \infty$.
  (a) Show that f is Bloch iff it is a Lipschitz map from the disk with the hyperbolic metric to the plane with its Euclidean metric.
  (b) Show that every bounded analytic function on $\mathbb{D}$ is in the Bloch class.
  (c) Show that $f(z) = \sum_{n=1}^{\infty} z^{2^n}$ is an unbounded function in the Bloch class. (Hint: fix z and estimate $f'(z)$ by splitting the sum for f' at N where $1 - |z| \simeq 2^{-N}$.)

(5) Suppose $\gamma \subset \mathbb{C}$ is a closed Jordan curve. Suppose f is continuous and bounded on $\mathbb{C}$ and is analytic on $\mathbb{C} \setminus \gamma$. Use the argument principle to show that $f(\gamma) = f(\mathbb{C})$. In other words, every value that f takes anywhere, is also taken on the curve. If f is non-constant, this means $f(\gamma)$ covers an open set. (Such a non-constant f exists whenever the curve is fractal in the sense that it has no tangent points, but this is not very easy to prove, e.g., Bishop, 1989).