**MAT 536, Spring 2024**
**PROBLEM SET 4, Due Monday, February 19**

Problems are due in class on Mondays (if you can't attend class, email solutions to me and the grader before class begins). Please restate problem at each beginning of your solution, or attach this sheet to the front of your solution set. Solutions should be legible and written in complete, correct sentences. Handwritten or a PDF from LaTeX is acceptable. Cite any sources that you use, e.g., a Wikipedia article or another textbook (most problems will not require outside sources).

(1) Prove there is a "universal" entire function $f$ with the property that given any other entire function $g$, there is a sequence of integers so that $f(z - n_k) \to g(z)$ uniformly on compact sets. (Hint: use Runge's theorem and the fact that any entire function can be approximated uniformly on compact sets by polynomials.)

(2) If $\{f_n\}_1^\infty$ are analytic on $\mathbb{C}$ and $f_n(z) \to f(z)$ for every $z \in \mathbb{C}$, must $f$ be analytic on $\mathbb{C}$? Prove or give a counterexample.

(3) Prove Goursat's theorem: if $f$ has a derivative at every point of $\Omega$ (but we don't assume $f'$ is continuous), then $f$ is holomorphic on $\Omega$. (Hint: follow the outline given in Exercise 4.12 of the textbook.)

(4) If $f$ is continuous on $\mathbb{C}$ and analytic on $\mathbb{C} \setminus \mathbb{R}$, then prove $f$ is analytic on all of $\mathbb{C}$. This says that the line is removable for continuous analytic functions.

(5) Suppose $\mu$ is a finite measure supported on a closed subset $X \subset \mathbb{C}$. Prove that

$$f(z) = \int_X \frac{d\mu(w)}{z - w},$$

is analytic on $\mathbb{C} \setminus X$ and give an integral formula for $f'$.

(Note: It is often possible to make sense of the integral defining $f$ even when $z \in X$. When $X = \mathbb{R}$, this is the Hilbert transform, and generalzing this to other sets and measures is a fundamental problem in the study of singular integral operators and geometric measure theory. )