# MAT 536, Spring 2024
# PROBLEM SET 3, Due Monday, February 12

Problems are due in class on Mondays (if you can't attend class, email solutions to me and the grader before class begins). Please restate problem at each beginning of your solution, or attach this sheet to the front of your solution set. Solutions should be legible and written in complete, correct sentences. Handwritten or a PDF from LaTeX is acceptable. Cite any sources that you use, e.g., a Wikipedia article or another textbook (most problems will not require outside sources).

(1) Prove that an injective analytic map $f : \mathbb{D} \to \mathbb{D}$ is surjective iff $f(z) = c(z-a)/(1-\overline{a}z)$, where $a \in \mathbb{D}$ and $c \in \mathbb{T}$.

(2) Suppose $U, V \subset \mathbb{C}$ are open, and that $f : U \to V$ is conformal (analytic, injective and surjective). Prove that $f$ is proper (the inverse image of any compact subset of $V$ is compact).

(3) Show that an analytic $f : \mathbb{D} \to \mathbb{D}$ is proper if and only if it has the form

$$f(z) = c \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a_k} z},$$

where $a_1, \ldots a_n \in \mathbb{D}$ and $c \in \mathbb{T}$. This is called a finite Blaschke product.

(4) The **pseudohyperbolic distance** on $\mathbb{D}$ is defined by $\rho(z,w) = \left|\frac{z-w}{1-\bar{w}z}\right|$.

The **hyperbolic metric** on $\mathbb{D}$ is given by $\delta(z,w) = \frac{1}{2}\log\left(\frac{1+\rho(z,w)}{1-\rho(z,w)}\right)$.

(a) Prove that if $\tau$ is as in Problem 1, then $\rho(z,w) = \rho(\tau(z), \tau(w))$.
(b) Prove the identity $1 - \rho^2(z,w) = (1-|z|^2)(1-|w|^2)/|1-\overline{w}z|^2$.
(c) Prove $\rho(|z|,|w|) \le \rho(z,w) \le \rho(|z|,-|w|)$.
(d) Prove $\rho$ and $\delta$ are metrics on $\mathbb{D}$.
(e) Suppose $f$ is analytic on $\mathbb{D}$ and $|f(z)| \le 1$. Prove $\rho(f(z), f(w)) \le \rho(z,w)$ and deduce the same fact for the hyperbolic metric.