MAT 536, Spring 2024
PROBLEM SET 2, Due Monday, February 5

Problems are due in class on Mondays (if you can't attend class, email solutions to me and the grader before class begins). Please restate problem at each beginning of your solution, or attach this sheet to the front of your solution set. Solutions should be legible and written in complete, correct sentences. Handwritten or a PDF from LaTeX is acceptable. Cite any sources that you use, e.g., a Wikipedia article or another textbook (most problems will not require outside sources).

(1) Prove that f has a power series expansion around z_0 of radius $r > 0$ if and only if $g(z) = (f(z) - f(z_0))/(z - z_0)$ has a power series expansion of radius r around z_0.

(2) Define $e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Show
   (a) this series converges for all $z \in \mathbb{C}$
   (b) $e^z e^w = e^{z+w}$
   (c) Define $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$, so that $e^{i\theta} = \cos\theta + i\sin\theta$. Show the series expansions for sin and cos are the same as from calculus.
   (d) $|e^z| = e^{\mathrm{Re}\, z}$ and $\arg e^z = \mathrm{Im}(z)$.
   (e) $e^z = 1$ only when $z = 2\pi k i$ for some integer k.
   (f) $\frac{d}{dz} e^z = e^z$.

(3) Suppose $\sum_{j=0}^{\infty} |a_j|^2 < \infty$. Show $f(z) = \sum_{j=0}^{\infty} a_j z^j$ is analytic in $\{z : |z| < 1\}$. Compute the following (in terms of the a_j's) and prove your answer:

$$\lim_{r \nearrow 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi}.$$

(4) Suppose f is analytic in a connected open set U such that for each $z \in U$, there exists an n (depending upon z) such that $f^{(n)}(z) = 0$. Prove f is a polynomial.

(5) Prove that the critical points of polynomial p are always contained within the convex hull of its roots. A critical point z is a point where $p'(z) = 0$. The convex hull of a set is the intersection of all closed half-planes containing the set.