MAT 536, Spring 2024
PROBLEM SET 1, Due Monday, January 29

Problems are due in class on Mondays (if you can't attend class, email solutions to me and the grader before class begins). Please restate problem at each beginning of your solution, or attach this sheet to the front of your solution set. Solutions should be legible and written in complete, correct sentences. Handwritten or a PDF from LaTeX is acceptable. Cite any sources that you use, e.g., a Wikipedia article or another textbook (most problems will not require outside sources).

(1) Find the three roots of $2x^3 - 9x^2 - 6x + 3 = 0$ using Cardano's formula. Which roots are real?

(2) The Jarkovski map is defined as $w(z) = \frac{1}{2}(z + 1/z)$. Prove the following facts:
  (a) w maps the unit circle $\mathbb{T} = \{z : |z| = 1\}$ to the interval $[-1, 1]$ and for points on the circle $w(z) = w(\overline{z}) = \text{Re}(z)$.
  (b) w maps $\{z : |z| > 1\}$ homeomorphically to $\mathbb{C} \setminus [-1, 1]$

(3) Let $\mathbb{S} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere. Show that any rigid rotation of $\mathbb{S}$ corresponds, via the stereographic projection, to a linear fractional transformation (LFT) of the plane, i.e., a map of the form $(az + b)/(cz + d)$. Give an example of an LFT which does not correspond to a rigid rotation of $\mathbb{S}$. (Hint: show the group of rotations of $\mathbb{S}$ is generated by the rotations around the three coordinate axes and that the non-constant LFTs form a group.)

(4) Show that for a continuous, complex valued function f on $[0, 2\pi]$,
$$\left| \frac{1}{2\pi} \int_0^{2\pi} f(t)dt \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(t)|dt$$
with equality iff f has constant argument.