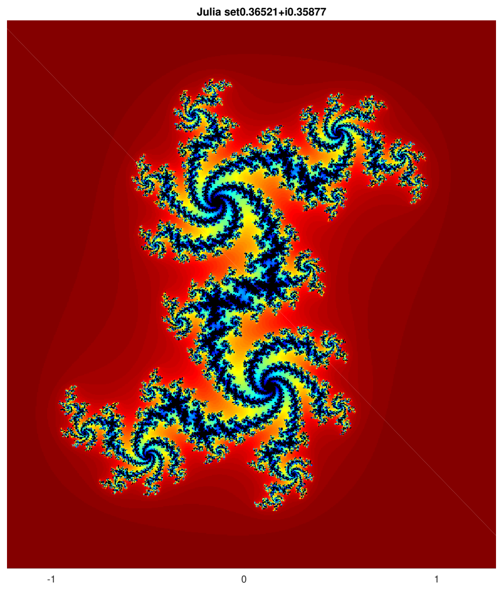
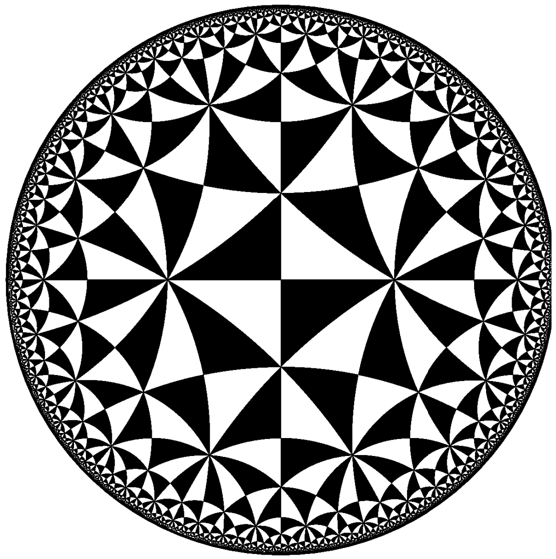
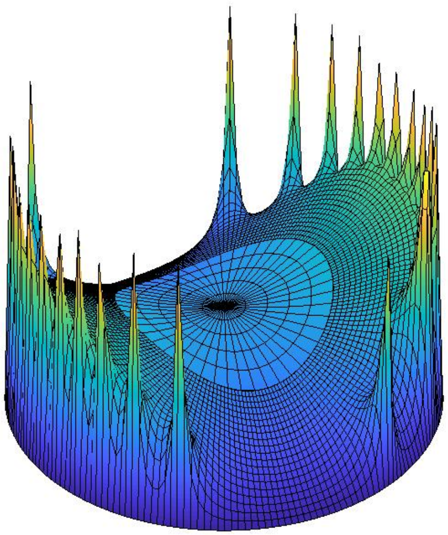
MAT 536, Spring 2024, Stony Brook University

Complex Analysis I, Christopher Bishop

Mon. Feb 5, 2024







Chapter 3: The maximum principle

Section 3.1: The maximum principle

Theorem 3.1 (The Maximum Principle): *Suppose f is analytic in a region Ω. If there exists a $z_0 \in \Omega$ such that*

$$|f(z_0)| = \sup_{z \in \Omega} |f(z)|$$

then f is constant in Ω.

Proof 1 (sketch):

Modify the "walking-the-dog" lemma from polynomials to non-constant analytic functions to show that for some k and $a_n \neq 0$,

$$f(z) - f(z_0) = a_k(z - z_0)^k + o(|z - z_0|^k).$$

This shows that near z_0, $|f|$ takes larger values than $|f(z_0)|$. $\square$

Proof 2:

Proof. We first note that for $n \geq 1$ and $z = z_0 + re^{it}$

$$\int_0^{2\pi} (z - z_0)^n dt = \int_0^{2\pi} re^{int} dt = r \int_0^{2\pi} (\cos nt + i \sin nt) dt = 0.$$

Thus

$$\int_0^{2\pi} f(z) dt = \int_0^{2\pi} [\sum_{n=0}^{\infty} a_n (z - z_0)^n] dt = \sum_{n=0}^{\infty} a_n [\int_0^{2\pi} (z - z_0)^n dt] = 2\pi a_0.$$

so

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z) dt.$$

This is the "mean value property".

Now suppose $|f|$ attains a maximum at z_0. Then with $z = z_0 + re^{it}$,

$$\begin{aligned} |f(z_0| &= |\frac{1}{2\pi}\int_0^{2\pi} f(z)dt| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} |f(z)|dt \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|dt \\ &\leq |f(z_0)| \end{aligned}$$

Therefore all the inequalities are equalities.

It is an exercise to show that if f is continuous on $[0, 2\pi]$, then

$$|\int_0^{2\pi} f(z)dt| = \int_0^{2\pi} |f(z)|dt$$

only occurs if f has constant argument (exercise). Assume $f > 0$.

But if f is not everywhere equal to $|f(z_0)|$, then

$$|\int_0^{2\pi} f(z)dt| < \int_0^{2\pi} |f(z_0)|dt = |f(z_0)|$$

which is a contradiction.

Thus $f(z)$ is constant on each small enough circle around z_0, and the mean value property implies this constant must be $f(z_0)$. $\square$

This proof works in many situtations, e.g., on manifolds and even on graphs (f is harmonic if the value at any vertex is the average over adjacent vertices).

Corollary 3.2: *If f is a non-constant analytic function in a bounded region Ω and if f is continuous on $\overline{\Omega}$ then*

$$\max_{z \in \overline{\Omega}} |f(z)|$$

occurs on $\partial\Omega$ but not in Ω.

$\partial\Omega$ = boundary of Ω.

Corollary: *if f is analytic on Ω then*

$$\limsup_{z\to\partial\Omega} |f(z)| = \sup_{\Omega} |f(z)|.$$

We say that a sequence tends to $\partial\Omega$ if it is eventually outside each compact subset of Ω. The lim sup is then the largest sub-sequential limit of the values of $|f|$ over all sequences that are eventually outside of each compact subset of Ω.

Exercise 2.6: Define $e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Show

(1) this series converges for all $z \in \mathbb{C}$

(2) $e^z e^w = e^{z+w}$

(3) Define $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$, so that $e^{i\theta} = \cos\theta + i\sin\theta$. Using the series for e^z show that you obtain the same series expansions for sin and cos that you learned in calculus. Check that $\cos^2\theta + \sin^2\theta = 1$, by multiplying out the definitions, so that $e^{i\theta}$ is a point on the unit circle corresponding to the Cartesian coordinate $(\cos\theta, \sin\theta)$.

(4) $|e^z| = e^{\mathrm{Re} z}$ and $\arg e^z = \mathrm{Im(z)}$. If z is a non-zero complex number then $z = re^{it}$, where $r = |z|$ and $t = \arg z$. Moreover, $z^n = r^n e^{int}$.

(5) $e^z = 1$ only when $z = 2\pi k i$ for some integer k.

Consequence: The function $f(z) = e^{-iz}$ is analytic in the upper half-plane $\mathbb{H} = \{z : \mathrm{Im(z)} > 0\}$, continuous on $\{z : \mathrm{Im(z)} \geq 0\}$ and has absolute value 1 on the real line $\mathbb{R}$, but is not bounded by 1 in $\mathbb{H}$.

The proof of the maximum principle applies in many circumstances:

- Harmonic functions in $\mathbb{R}^n$ satisfy the maximum principle (Chapter 7).

- Sub-harmonic functions satisfy the maximum principle (Chapter 7).

- Analytic functions on a compact Riemann surface are constant (Chap 14).

- If a function f on the vertices of a connected graph is always the average of its values on adjacent vertices, then it is constant.

Section 3.2: Local behavior

f is called an **open map**, if the image of every open set is open. $U \subset \Omega$ is open then $f(U)$ is an open set.

(Recall f is continuous if inverse image of open sets are open. $f(x) = x^2$ is continuous but not open on $\mathbb{R}$).

We foreshadowed the following with the "walking-the-dog" lemma.

Corollary 3.3: *A non-constant analytic function defined on a region is an open map.*

Proof. Suppose f has a power series expansion which converges on $D(z_0, r) = \{z : |z - z_0| < R\}$. Pick $r < R$ and set

$$\delta = \inf_{|z-z_0|=r} |f(z) - f(z_0)|.$$

Since the zeros of $f - f(z_0)$ are isolated by Corollary 2.9, we may suppose that $\delta > 0$ by decreasing r if necessary.

If $|w - f(z_0)| < \delta/2$ and if $f(z) \neq w$ for all z such that $|z - z_0| \leq r$, then $1/(f - w)$ is analytic in $|z - z_0| \leq r$ and on $|z - z_0| = r$ we have

$$\left|\frac{1}{f(z) - w}\right| \leq \frac{1}{|f(z) - f(z_0)| - |w - f(z_0)|} < \frac{1}{\delta - \delta/2} = \frac{2}{\delta}.$$

By the maximum principle the inequality $|1/(f(z) - w)| < 2/\delta$ persists in $|z - z_0| < r$. But evaluating this expression at z_0 we obtain the contradiction $2/\delta < 2/\delta$.

Thus the image of the disk of radius r about z_0 contains a disk of radius $\delta/2$ about $f(z_0)$.

This implies that the image of an open set contains a neighborhood of each of its points. $\square$

Definition: A function f is **one-to-one** if $f(z) = f(w)$ only when $z = w$.

This is also called "injective".

Marshall says "A more enlightened terminology proposed by one of my teachers, Richard Arens, is **two-to-two** since two points go to two points. "One-to-one" should really be the definition of a function, but the inertia of common usage is too large to overcome."

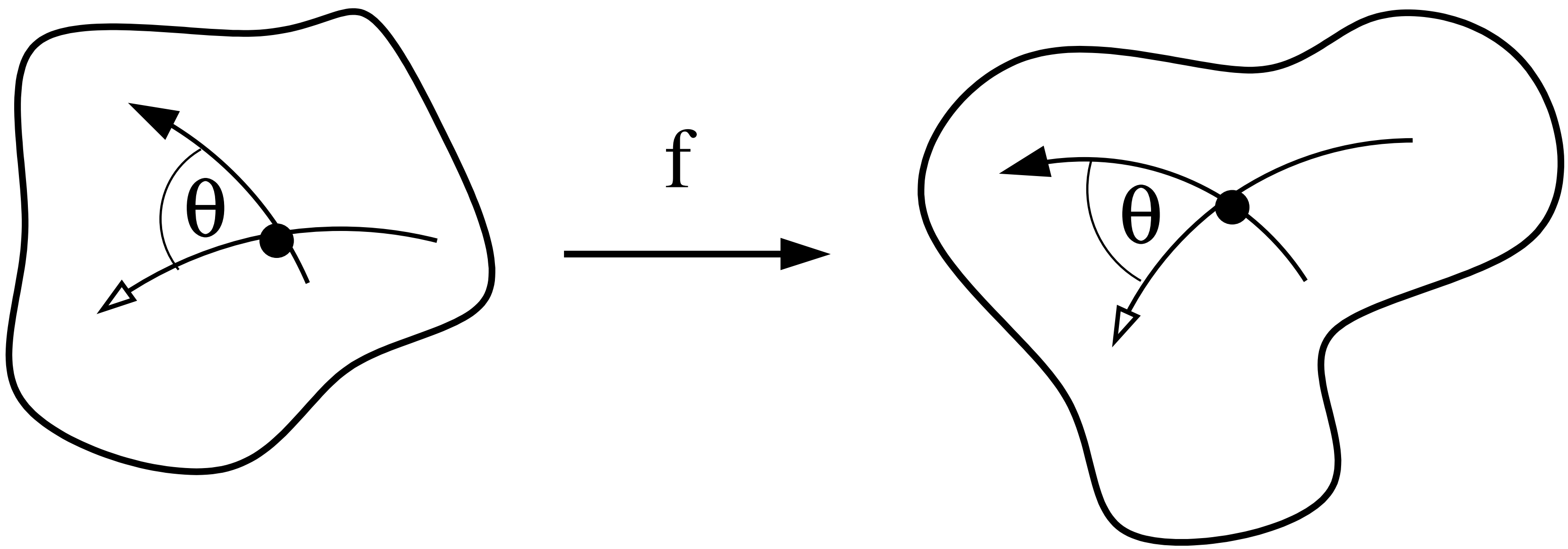
See Richard Arens. He was my great-great-grand-advisor (Garett Birkhoff $\to$ Arens $\to$ Glicksberg $\to$ Garnett $\to$ Jones $\to$ Bishop.

Math Genealogy

Proposition 3.5: *If f is analytic at z_0 with $f'(z_0) \neq 0$, then f is one-to-one in a sufficiently small neighborhood of z_0.*

Proof. By Cor 2.14, if $z_n, w_n \to z_0$ with $f(z_n) = f(w_n)$ then $f'(z_0) = 0$. $\square$

Definition: We say that f is **locally conformal** if it preserves angles (including direction) between curves.



Corollary 3.7 *Suppose f is analytic at z_0. Then*

$$f(z) - f(z_0) = \sum_{m=n}^{\infty} a_m (z - z_0)^m$$

with $a_n \neq 0$ if and only if for ϵ sufficiently small, there exists $\delta > 0$ so that $f(z) - w$ has exactly n distinct roots in $\{z : 0 < |z - z_0| < \epsilon\}$, provided $0 < |w - f(z_0)| < \delta$.

Proof uses two exercises from the textbook.

Analytic f preserves angles except at critical points, where angles are multiplied by an integer.

Exercise 2.5: Prove that f has a power series expansion about z_0 with radius of convergence $r > 0$ if and only if

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0}$$

has a power series expansion about z_0, with the same radius of convergence.

Exercise 2.10: Let n be a positive integer. Prove that $z^{\frac{1}{n}}$ is analytic in $B = \{z : |z-1| < 1\}$ in the following sense: there is a convergent power series f in B with the property that $f(z)^n = z$ and $f(1) = 1$.

Hint: write $z = 1 + w$, $|w| < 1$, and let $g(w) = \sum a_k w^k$ be the (formal) Taylor series for $(1+w)^{\frac{1}{n}}$. Then prove $|a_k| \leq 1/k$ so that g is analytic in $|w| < 1$. Use Taylor's theorem to show that $g(x) = (1+x)^{1/n}$ for $-1 < x < 1$, and then use the uniqueness theorem, Corollary 4.4, to show that $g(w)^n = 1 + w$.

Alternatively, you can prove that $z^{1/n}$ can be defined so that it has derivatives of all orders and prove that Taylor's theorem is true for complex differentiable functions using complex integration.

Proof. By repeated division as in Exercise 2.5, if f is analytic at z_0, we can write

$$f(z) - f(z_0) = a_n(z - z_0)^n g(z),$$

where a_n is the first non-zero power series coefficient after a_0 and g is analytic at z_0 with $g(z_0) = 1$.

Choose a so that $a^n = a_n$. By Exercise 2.10, we can define $z^{\frac{1}{n}}$ to be analytic in a neighborhood of 1. Set $F(z) = g(z)^{\frac{1}{n}}$. Then

$$f(z) = f(z_0) + [a(z - z_0)F(z)]^n.$$

By Proposition 3.5, $a(z - z_0)F(z)$ is one-to-one in a neighborhood of z_0. $\square$

Section 3.3: Growth on $\mathbb{C}$ and $\mathbb{D}$

Corollary 3.8, Liouville's Theorem: *If f is analytic in $\mathbb{C}$ and bounded, then f is constant.*

Proof. Suppose $|f| \leq M < \infty$. Set $g(z) = (f(z) - f(0))/z$. Then g is analytic and $|g| \to 0$ as $|z| \to \infty$. By the maximum principle $g \equiv 0$ and hence $f \equiv f(0)$. $\square$

Corollary: *If f is analytic in $\mathbb{C}$ and if $|f(z)| = O(|z|^n)$ as $|z| \to \infty$, then f is a polynomial.*

Proof. Let $p(z) = \sum_{k=0}^{n} a_k z^k$ be the terms of the power series expansion of f at 0 up to degree n.

Then $g(z) = (f(z) - p(z))/z^n$ is analytic in $\mathbb{C}$ and bounded. By Liouville's theorem, g is constant and hence f must be a polynomial. In fact, $f = p$ since $g(0) = 0$. $\square$

Corollary 3.9, Schwarz's Lemma: *Suppose* f *is analytic in* $\mathbb{D}$ *and suppose* $|f(z)| \leq 1$ *and* $f(0) = 0$. *Then*

$$|f(z)| \leq |z|, \tag{3.7}$$

for all $z \in \mathbb{D}$ *and*

$$|f'(0)| \leq 1. \tag{3.8}$$

Moreover, if equality holds in (3.7) for some $z \neq 0$ *or if equality holds in (3.8) then* $f(z) = cz$ *where* c *is a constant with* $|c| = 1$.

Exercise 2.5: Prove that f has a power series expansion about z_0 with radius of convergence $r > 0$ if and only if

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0}$$

has a power series expansion about z_0, with the same radius of convergence.

Proof. By Exercise 2.5, the function g given by

$$g(z) = \begin{cases} \dfrac{f(z)}{z}, & \text{if } z \in \mathbb{D} \setminus \{0\} \\ \\ f'(0) & \text{if } z = 0 \end{cases}$$

is analytic in $\mathbb{D}$ and for $0 < r < 1$

$$\sup_{|z|=r} |g(z)| \leq \frac{1}{r}.$$

Fix $z_0 \in \mathbb{D}$, then for $r > |z_0|$, the maximum principle implies $|g(z_0)| \leq \frac{1}{r}$, so that, letting $r \to 1$, we obtain (3.7) and (3.8). If equality holds in (3.7) at z_0 or holds in (3.8) then $g(z)$ has a maximum at z_0 or 0 and hence is constant. $\square$

A linear fractional transformation is a (non-constant) map of the form

$$f(z) : \frac{az+b}{cz+b}.$$

- Also called “Möbius transformations”.

- Is 1-1 map of 2-sphere to itself (only such).

- Map circles/lines to circles/lines (Theorem 6.8).

- Determined by images of any three distinct points.

- Form a group = 6-dimensional Lie group $\mathrm{PSL}(2,\mathbb{C})$.

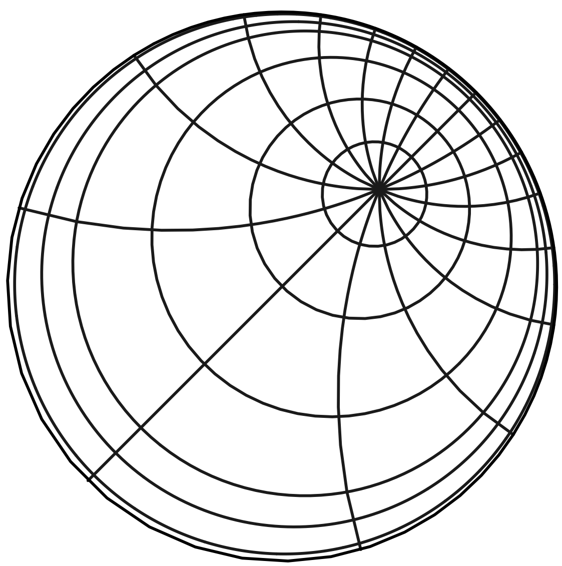
Lemma: $T_c(z) = (z - c)/(1 - \overline{c}z)$ maps unit disk 1-1 onto unit disk.

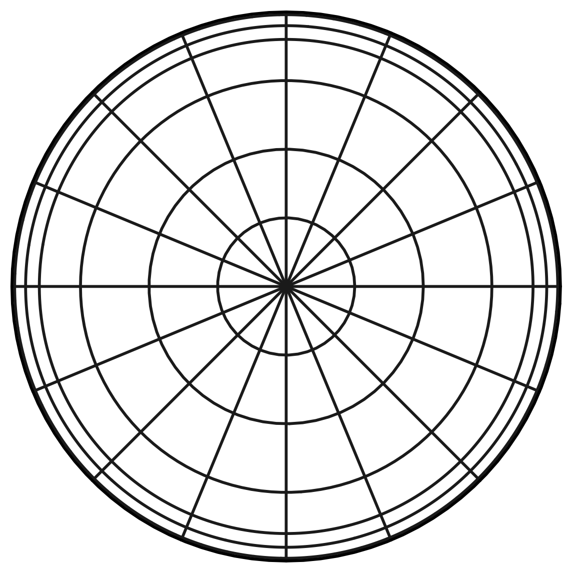
Proof. T_c is analytic except at pole $1/\overline{c}$, outside $\mathbb{D} = \{|z| < 1\}$. For $z = e^{it} \in \mathbb{T}$,

$$|T_c(z)| \left| \frac{z - c}{1 - \overline{c}z} \right| = \left| \frac{e^{it} - c}{1 - \overline{c}e^{it}} \right| = \left| \frac{e^{it} - c}{e^{-it} - \overline{c}} \right| = 1.$$

So T_c maps unit circle into itself. By maximum principle, maps $\mathbb{D} \to \mathbb{D}$.

$T_{-c} = (T_c)^{-1}$ has same properties, so T_c is 1-1, onto from $\mathbb{D}$ to $\mathbb{D}$. $\square$





Corollary 3.10, Invariant form of Schwarz's lemma: *Suppose* f *is analytic in* $\mathbb{D} = \{z : |z| < 1\}$ *and suppose* $|f(z)| < 1$*. If* $z, a \in \mathbb{D}$ *then*

$$\left|\frac{f(z) - f(a)}{1 - \overline{f(a)}f(z)}\right| \leq \left|\frac{z - a}{1 - \overline{a}z}\right|, \tag{3.9}$$

and

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}. \tag{3.10}$$

Proof. Setting $c = f(a) \in \mathbb{D}$, and using Exercise 2.5 again,

$$\frac{T_c \circ f(z)}{T_a(z)} = \left(\frac{f(z) - f(a)}{1 - \overline{f(a)} f(z)}\right)\left(\frac{1 - \overline{a}z}{z - a}\right)$$

is analytic on $\mathbb{D}$.

Moreover,

$$\limsup_{|z| \to 1} \left|\frac{T_c \circ f(z)}{T_a(z)}\right| = \limsup_{|z| \to 1} |T_c \circ f(z)| \leq 1.$$

So by the maximum principle, (3.9) holds.

(3.10) follows by dividing both sides of (3.9) by $|z - a|$ and letting $z \to a$. $\square$

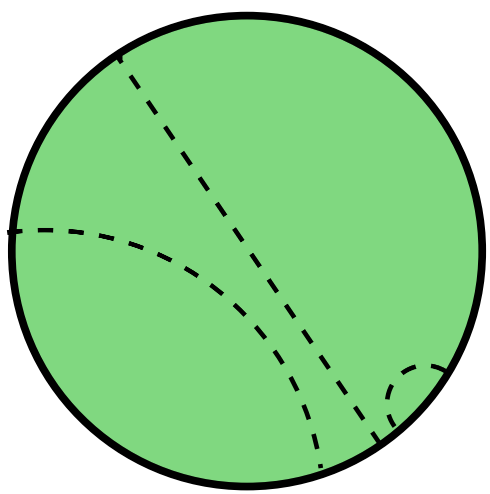
Hyperbolic geometry: We define the **quasi-hyperbolic metric** on $\mathbb{D}$ by

$$\widetilde{\rho}(z,w) = \left|\frac{z-w}{1-\overline{w}z}\right|.$$

• The invariant form of Schwarz's lemma says that any analytic map $f : \mathbb{D} \to \mathbb{D}$ is a contraction for this metric.

• Linear fractional maps $\mathbb{D} \to \mathbb{D}$ are isometries for this metric.

• Geodesics are circles orthogonal to the boundary, lines through the origin.



The **hyperbolic metric** can be defined as

$$\rho(z,w) = \frac{1}{2}\log\frac{1+\widetilde{\rho}(w,z)}{1-\widetilde{\rho}(w,z)}.$$

This implies analytic maps also contract this metric, and Möbius maps of $\mathbb{D} \to \mathbb{D}$ are isometries.

More standard definition is to take

$$\rho(z,w) = \inf_{\gamma}\int_{\gamma}\frac{ds}{1-|z|^2},$$

where the infinitum is over all rectifiable paths in $\mathbb{D}$ connecting z to w.

See Exercise 3.9.

The pseudo-hyperbolic metric can also be defined as

$$\widetilde{\rho}(z, w) = \sup\{|f(z) - f(w)| : f \text{ is analytic on } \mathbb{D}, |f| \leq 1 \text{ on } \mathbb{D}\}.$$

This makes sense on any manifold where analytic functions are defined.

Called the **Kobayashi metric**.

What is the Kobayashi metric on the complex plane?

Corollary 3.11: *If f is analytic on $\mathbb{D}$, $|f| \leq 1$, and $f(z_j) = 0$, for $j = 1, \ldots, n$ then*

$$f(z) = \prod_{j=1}^{n} \left(\frac{z - z_j}{1 - \overline{z_j} z} \right) g(z),$$

where g is analytic in $\mathbb{D}$ and $|g(z)| \leq 1$ on $\mathbb{D}$.

A function of the form

$$B(z) = \prod_{j=1}^{n} \left(\frac{z - z_j}{1 - \overline{z_j} z} \right)$$

is called a **finite Blaschke product**.

• Finite Blaschke products are exactly the analytic maps on $\mathbb{D}$ that extend continuously to the boundary $\mathbb{T}$ and map $\mathbb{T}$ into itself.

• They are also exactly the proper, analytic maps from $\mathbb{D}$ to $\mathbb{D}$.

Defn: proper means the inverse image of a compact set is compact.

• Infinite Blaschke products can exist, but there is a restriction on $\{z_j\}$.

Corollary 3.12: *If f is non-constant, bounded, and analytic in $\mathbb{D}$ and if $\{z_j\}$ are the zeros of f then*

$$\sum_j (1 - |z_j|) < \infty.$$

The convention we adopt here is that if z_j is a zero of order k, then $(1 - |z_j|)$ occurs k times in the sum in the statement of Corollary 3.12.

Proof. We may suppose $|f| \leq 1$, by dividing f by a constant if necessary.

If $f(0) \neq 0$, then using the notation of the proof of Corollary 3.11,

$$|f(0)| = (\prod_{j=1}^{n} |z_j|)|g(0)| \leq \prod_{j=1}^{n} |z_j|,$$

so that by taking logarithms (base e),

$$\log \frac{1}{|f(0)|} \geq \sum_{j=1}^{n} \log \frac{1}{|z_j|} \geq \sum_{j=1}^{n} (1 - |z_j|).$$

If $f(0) = 0$, then write $f(z) = z^k h(z)$ where $h(0) \neq 0$. Applying the preceding argument to h, we obtain

$$\sum_{j=1}^{n} (1 - |z_j|) \leq \log \frac{1}{|h(0)|} + k.$$

The corollary follows by letting $n \to \infty$. $\square$

Theorem 11.19: *If* $\{z_j\} \subset \mathbb{D}$ *and* $\sum_j (1 - |z_j|) < \infty$, *then*

$$B(z) = \prod_{j=1}^{n} \frac{|z_j|}{-z_j} \left(\frac{z - z_j}{1 - \overline{z_j} z} \right)$$

defines an analytic function of $\mathbb{D}$ *that has zeros exactly at the points* $\{z_j\}$.

See Chapter 11 for the proof.

Section 3.4: Boundary behavior

Exercise 2.6: Define $e^z = exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Show

(1) this series converges for all $z \in \mathbb{C}$

(2) $e^z e^w = e^{z+w}$

(3) Define $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$, so that $e^{i\theta} = \cos\theta + i\sin\theta$.

(4) $|e^z| = e^{\mathrm{Re} z}$ and $\arg e^z = \mathrm{Im(z)}$. If z is a non-zero complex number then $z = re^{it}$, where $r = |z|$ and $t = \arg z$. Moreover, $z^n = r^n e^{int}$.

(5) $e^z = 1$ only when $z = 2\pi ki$ for some integer k.

Consequence: By (4), e^z maps that left half-plane into $\mathbb{D} \setminus \{0\}$.

Claim: The linear fractional map

$$\tau(z) = \frac{z+1}{z-1}$$

maps the unit disk to the left half-plane.

- Check that $\tau(-1) = 0$, $\tau(1) = \infty$, $\tau(i) = (i+1)/(i-1) = i$.

- Since τ maps circles to circles/lines, it maps unit circle to imaginary axis (Theorem 6.3).

- Since $\tau(0) = 0$, the disk maps to the left-half-plane.

- $\lim_{x \to -1} \tau(x) = -\infty$.

Hence $f(z) = \exp(\frac{z+1}{z-1})$ maps $\mathbb{D}$ to $\mathbb{D} \setminus \{0\}$.

- This is an ∞-to-1 map.

- Maps $\mathbb{T} \setminus \{1\}$ into $\mathbb{T}$.

- But $\lim_{x \to 1} f(x) = 0$.

This is an example of an *inner function*, a bounded analytic function S so that $|S|$ has radial limits 1 almost everywhere. Blaschke products (finite or infinite) are also examples.

An inner function is called singular if it has no zeros, as above.

Inner functions play an important role in functional analysis, e.g., operators on Hilbert spaces. See Beurling's theorem.

The series

$$g(z) = \sum_{n=1}^{\infty} z^{n!}$$

converges on $\mathbb{D}$ and defines an analytic function.

If p/q is a rational number, then $(p/q) \cdot n!$ is an integer for $n \geq q$.

Thus if $w = \exp(2\pi i \frac{p}{q})$, then $w^n = 1$ for $n \geq q$. Therefore,

$$\lim_{r \nearrow 1} g(rw) = \lim_{r \nearrow 1} \sum_{n=1}^{q-1} w^{n!} + \lim_{r \nearrow 1} \sum_{n=q}^{\infty} r^n = O(1) + \lim_{r \nearrow 1} \frac{r^q}{1-r} = \infty.$$

Thus $g \to \infty$ along a dense set of radii.

With more work, we can show g fails to have a radial limit Lebesgue almost everywhere on $\mathbb{T}$. However, it stays bounded on a set of radii whose endpoints on $\mathbb{T}$ have Hausdorff dimension 1 (a large set of zero length).

See The boundary behavior of Bloch functions by S. Rohde.

Fatou's theorem: If f is bounded and analytic on $\mathbb{D}$, then

$$\lim_{r \nearrow 1} f(re^{it})$$

exists for almost every t. "Radial limits exist a.e."

This is not given in our textbook, but it can be proven using, in part, the Hardy-Littlewood maximal theorem from MAT 532 (Real Analysis I).

It is possible for a bounded, analytic function not to extend continuously to any boundary point. For, example, this happens for any infinite Blaschke product whose zeros accumulate everywhere on $\mathbb{T}$. (Every boundary point is a limit of zeros, but almost every radial limit value has absolute value 1.)

The function

$$h(z) = \sum_{n=1}^{\infty} \frac{z^{2^n}}{n^2}$$

is analytic on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$ but $h(\mathbb{T})$ has interior.

In other words, it defines a Peano curve on $\mathbb{T}$. Thus analytic functions have have quite wild boundary values (even when they are continuous).

See Lacunary power series and Peano curves by R. Salem and A. Zygmund.

The Salem Prize

Calderon-Zygmund theory