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REAL ANALYSIS II

FOLLAND'S REAL ANALYSIS: CHAPTER 9
ELEMENTS OF DISTRIBUTION THEORY

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Chapter 9: Elements of Distribution Theory

9.1 Distributions

9.2 Compactly Supported, Tempered and Periodic Distributions

9.3 Sobolev Spaces

Chapter 9.1: Distributions

$C_c^\infty(E)$ = compactly supported smooth functions with support inside E.

If U is open in $\mathbb{R}^n$ we define:

(i) A sequence $\{\phi_j\}$ in $C_c^\infty(U)$ **converges in** C_c^∞ to ϕ $\{\phi_j\} \subset C_c^\infty(K)$ for some compact set $K \subset U$ and $\phi_j \to \phi$ in the topology of $C_c^\infty(K)$, that is, $\partial^\alpha \phi_j \to \partial^\alpha \phi$ uniformly for all α.

(ii) If $\mathcal{X}$ is a locally convex topological vector space and $T : C_c^\infty(U) \to \mathcal{X}$ is a linear map, then T is continuous if for each compact $K \subset U$, T restricted to $C_c^\infty(K)$ is continuous, that is $T\phi_j \to T\phi$ whenever $\phi_j \to \phi$ in $C_c^\infty(K)$ and $K \subset U$ is compact.

(iii) A linear map $T : C_c^\infty(U) \to C_c^\infty(U')$ is continuous if for each compact $K \subset U$ there is a compact $K' \subset U'$ such that $T(C_c^\infty(K)) \subset C_c^\infty(K')$ and T is continuous from $C_c^\infty(K)$ to $C_c^\infty(K')$.

(iv) A **distribution** on U is a continuous linear functional on $C_c^\infty(U)$. The space of all distributions on U is denoted by $\mathcal{D}'(U)$, and we set $\mathcal{D}' = \mathcal{D}'(\mathbb{R}^n)$. We impose the weak* topology on $\mathcal{D}'(U)$, that is, the topology of pointwise convergence on $C_c^\infty(U)$.

$\mathcal{D}$ is Schwarz's notation for C_c^∞.

Examples of Distributions:

• Every $f \in L^1_{\text{loc}}$, i.e., every function f on U that is integrable on every compact set.

• Every Radon measure μ on U defines a distribution by $\phi \to \int \phi d\mu$. .

• If $x_o \in U$ and α is a multi-index, the map $\phi \to \partial^\alpha \phi(x_0)$ is a distribution. This does not arise from a function; it arises from a measure precisely when $\alpha = 0$, in which case it is the point mass at x_0.

Notation: If $F \in \mathcal{D}'(U)$ and and $\phi \in C_c^\infty(U)$, the value of F at ϕ, is denoted by $\langle F, \phi \rangle$. This is linear in each variable; this conflicts with our earlier notation for inner products, but will cause no serious confusion.

Sometimes it is convenient to pretend that a distribution is a function when it really is not, and to write $\int F(x)\phi(x)dx$ instead of $\langle F, \phi \rangle$. This is the case especially when the explicit presence of the variable x is notationally helpful.

Notation: We shall use a tilde to denote the reflection of function in the origin:

$$\tilde{\phi}(x) = \phi(-x).$$

Notation: We denote the point mass at the origin, by δ:

$$\langle \delta, \phi \rangle = \phi 0).$$

The following is an important corollary of Theorem 8.14:

Prop 9.1: Suppose that $f \in L^1(\mathbb{R}^n)$ and $\int f = 1$, and for $t > 0$ let $f_t(x) = t^{-n} f(x/t)$. Then $f_t \to a\delta$ in $\mathcal{D}'$ as $t \to 0$.

Proof. If $\phi \in C_c^\infty$ then by Theorem 8.14 we have

$$\langle f_t, \phi \rangle = \int f_y \phi = f_t * \tilde{\phi}(0) \to a\tilde{\phi}(0) = a\phi(O) = a\langle \delta, \phi \rangle. \quad \square$$

If does not make sense to say two distributions agree at a point.

We say two distributions on U agree on an open subset $V \subset U$ if they agree on all functions in $C_c^\infty(V)$.

Prop 9.2: Let $\{V_\alpha\}$ be a collection of open subsets of U and let $V = \cup V_\alpha$. If $F, G \in \mathcal{D}'(U)$ agree on every V_α then they agree on V.

Proof. If $\phi \in C_c^\infty(V)$, then it has compact support, so this support is contain in a finite union $V_{\alpha_1} \cup \cdots \cup V_{\alpha_m}$. Pick $\psi_1, \ldots, \psi_m \in C_c^\infty$ so that $\sum \psi_m = 1$ on $\mathrm{supp}(\phi)$. (This is the C^∞ analogue of Proposition 4.41). Then

$$\langle F, \phi \rangle = \sum \langle F, \psi_j \phi \rangle = \sum \langle G, \psi_j \phi \rangle = \langle G, \phi \rangle. \quad \square$$

According to Proposition 9.2, there is a maximal open subset V of U on which F agrees with the zero distribution. The complement in U of V is called the support of F.

There is a general procedure for extending various linear operations from functions distributions.

Suppose that U and V are open sets in $\mathbb{R}^n$, and T is a linear map from some subspace $\mathcal{X} \subset L^1_{\text{loc}}(U)$ into $L^1_{\text{loc}}(V)$.

Suppose that there is another linear map $T' : C_c\infty(V) \to C_c\infty(U)$ such that

$$\int (Tf)\phi = \int f(T'\phi), f \in \mathcal{X}, \phi \in C_c^\infty(V).$$

Then T can be extended to a map from $\mathcal{D}'(U) \to \mathcal{D}'(V)$ by

$$\langle TF, \phi \rangle = \langle F, T'\phi \rangle, \quad F \in \mathcal{D}'(U), \quad \phi \in C_c^\infty(U).$$

Examples:

i. (Differentiation): Let $Tf = \partial^\alpha f$, defined on $C^{|\alpha|}(U)$. If $\phi \in C_c^\infty(U)$, integration by parts gives

$$\int (\partial^\alpha f)\phi = (-1)^{|\alpha|} \int f(\partial^\alpha \phi).$$

(there are no boundary terms since ϕ has compact support.) Hence T' is the restriction of $(-1)^{|\alpha|}T$ to $C_c^\infty(U)$. We define the derivative of a distribution $F \in \mathcal{D}'(U)$ by

$$\langle \partial^\alpha F, \phi \rangle = (-1)^{|\alpha|} \langle F, \partial^\alpha \phi \rangle$$

In particular, we can define derivatives of any locally integrable functions or any finite measure.

ii. (Multiplication by smooth functions): Given $\psi \in C^\infty(U)$, define $Tf = \psi f$ and let T' be the restriction of T to $C_c^\infty(U)$. The for a distribution $F \in \mathcal{D}'(U)$ we define ψF by

$$\langle \psi F, \phi \rangle \langle F, \psi \phi \rangle.$$

iii. (Translation): Given $y \in \mathbb{R}^n$ let $V = U+ = \{x + y : x \in U\}$ and let $T = \tau_y$. Since

$$\int f(x-y)\phi(x)dx = \int f(x)\phi(x+y)dx$$

we have that T' is the restriction of τ_{-y} to $C_c^\infty(U+y)$. For a distribution $F \in \mathcal{D}'(U)$, we define the translated distribution $\tau_y F$ by

$$\langle \tau_y F, \phi \rangle = \langle F, \tau_{-y}\phi \rangle.$$

iv. (Composition with linear maps): Given an invertible linear transformation S on $\mathbb{R}^n$ let $V = S^{-1}(U)$ and let $Tf = f \circ S$. Then

$$T'\phi = \frac{\phi \circ S^{-1}}{|\det(S)|}.$$

$\tau_y F$ by

$$\langle F \circ S, \phi \rangle = \langle F, \phi \circ S^{-1} \rangle / |\det(S)|.$$

v. (Convolution, First Method): Given $\psi \in C_c^\infty(U)$, let

$$V = \{x : x - y \in U \text{ for } y \in \operatorname{supp}(\psi)\}.$$

(V is open but may be empty.) If $f \in L^1_{\text{loc}}(U)$, the integral

$$f * \psi(x) = \int f(x-y)\psi(y)dy = \int f(y)\psi 0 x_y)dy = \int f(\tau_y \tilde{\psi})$$

is well defined for all $x \in V$. The convolution of $F * \psi$ is the function defined on V by

$$F * \psi(x) = \langle F, \tau_x \tilde{\psi} \rangle.$$

This is a continuous function of x (actually C^∞; see below).

If $\psi \in C_c^\infty$ we have

$$\delta * \psi = \langle \delta, \tau_x \tilde{\psi} \rangle = \tau_x \tilde{\psi}(0) = \psi(x).$$

Thus δ is the multiplicative identity for convolution.

vi. (Convolution, Second Method): Let $\psi, \tilde{\psi}, V$ be as above. If $f \in L^1_{\text{loc}}(U)$ and $\phi \in C_c^\infty(V)$ then

$$\int (f * \psi)\phi = \iint f(y)\psi(x-y)\phi(y)dydx = \int f(\phi * \tilde{\psi}).$$

Hence convolution with ψ, $Tf = f*\psi$ maps $L^1_{\text{loc}}(U)$ to $L^1_{\text{loc}}(V)$. Let $T'\phi = \phi*\tilde{\psi}$. For a distribution $F \in \mathcal{D}'(U)$ we can define convolution with ψ as a distribution in $\mathcal{D}'(V)$ by

$$\langle F * \psi, \phi \rangle = \langle F, \phi * \tilde{\psi} \rangle.$$

This is a continuous function of x (actually C^∞; see below).

We will show the definitions of convolution in (v) and (vi) agree.

Prop 9.3: Suppose $U \subset \mathbb{R}^n$ is open and$\psi \in C_c^\infty$. Let

$$V = \{x : x - y \in U \text{ for } y \in \mathrm{supp}(\psi)\}.$$

For $F \in \mathcal{D}'(U)$ and $x \in V$ let

$$F * \psi(x) = \langle F, \tau_x \psi \rangle.$$

Then

(a) $F * \psi \in C^\infty(V)$.

(b) $\partial^\alpha (F * \psi) = (\partial^\alpha F) * \psi = F * (\partial^\alpha \psi)$.

(c) For any $\phi \in C_c^\infty(V)$, $\int (F * \psi)\phi = \langle F, \phi * \tilde{\psi} \rangle$.

Proof. Let $\{e_1, \ldots e_n\}$ be the standard basis of $\mathbb{R}^n$. Since V is open, if $x \in V$, then there is $t_0 > 0$ so that $x + te_j \in V$ for $|t| < t_0$. Then

$$\frac{1}{t}(\tau_{x+te_j}\tilde{\psi} - \tau_x\psi) \to \tau_x\widetilde{\partial_j\psi},$$

in $C_c^\infty(U)$ as $t \to 0$.

It follows that $\partial_j(F * \psi)$ exists and equals $F * \partial_j\psi(x)$. By induction we get $F * \psi \in C^\infty(V)$ and $\partial^\alpha(F * \psi) = F * (\partial^\alpha\psi)$. This proves (a).

Moreover, since

$$\partial^\alpha \tilde{\psi} = (-1)^{|\alpha|}\widetilde{\partial^\alpha \psi} \text{ and } \partial^\alpha \tau_x = \tau_x \partial^\alpha,$$

we have

$$(\partial^\alpha F)*\psi(x) = \langle \partial^\alpha F, \tau_x \tilde{\psi}\rangle = (-1)^{|\alpha|}\langle F, \partial^\alpha \tau_x \tilde{\psi}\rangle = \langle F, \tau_x \widetilde{\partial^\alpha \psi}\rangle = F*(\partial^\alpha \psi)(x).$$

This proves (b).

If $\phi \in C_c^\infty(V)$, then

$$\phi * \tilde{\psi}(x)) = \int \phi(y)\psi(y-x)dy = \int \phi(y)\tau_y\tilde{\psi}(x)dy.$$

The integrand is continuous and supported in a compact subset of U, so it can be approximated by Riemann sums. More precisely, approximate $\operatorname{supp}(\phi)$ by a union of cubes of side length 2^{-m} centered at points $y_1^m, \dots, y_{k(m)}^m$ in $\operatorname{supp}(\phi)$. The Riemann sums

$$S^m = 2^{-nm} \sum \phi(y_j^m)\tau_{y_j^m}\tilde{\psi},$$

are supported in a common compact subset of U and converge uniform to $\phi * \tilde{\psi}$ as $m \to \infty$.

Likewise

$$\partial^\alpha S^m = 2^{-nm} \sum \phi(y_j^m) \tau_{y_j^m} \partial^\alpha \tilde{\psi},$$

converges to

$$\phi * \partial^\alpha \tilde{\psi} = \partial^\alpha * \phi * \tilde{\psi}),$$

so $S^m \to \phi * \tilde{\psi}$ in $C_c^\infty(U)$. Hence

$$\begin{aligned}
\langle F, \phi * \tilde{\psi} \rangle &= \lim_{m\to\infty} \langle F, S^m \rangle \\
&= \lim_{m\to\infty} 2^{-nm} \sum \phi(y_j^m) \langle F, \tau_{y_j^m} \tilde{\psi} \rangle \\
&= \int \phi(y) \langle F, \tau_y \tilde{\psi} \rangle \\
&= \int \phi(y) F * \psi(y) dy \quad \square
\end{aligned}$$

Lemma 9.4: Suppose that $\phi, \psi \in C_c^\infty$ and $\int \psi = 1$ and let $\psi_t = t^{-n}\psi(x/t)$.

(a) Given any neighborhood U of $\mathrm{supp}(\phi)$, we have $\mathrm{supp}(\phi * \psi_t) \subset U$ for t sufficiently small.

(b) $\phi * \psi_t \to \psi$ in C_c^∞ as $t \to 0$.

Proof. If $\mathrm{supp}(\psi) \subset B(0, R)$ then $\mathrm{supp}(\phi * (\psi_t)$ is contained in a tR neighborhood of $\mathrm{supp}(\phi)$, which is in U for t small. . Moreover,

$$\partial^\alpha(\phi * \psi_t) = (\partial^\alpha \phi) * \psi_t \to \partial^\alpha \phi$$

uniformly as $t \to 0$ By Theorem 8.14. $\square$

Prop. 9.5: For any open $U \subset \mathbb{R}^n$, $C_c^\infty(U)$ is dense in $\mathcal{D}'(U)$.

Proof. Suppose $F \in \mathcal{D}'(U)$. We shall first approximate F by distributions supported in compact subsets of U, then approximate the latter by functions in $C_c^\infty(U)$.

Let $\{V_j\}$ be an increasing sequence of precompact open subsets of U whose union is U, as in Proposition 4.39. For each j, by the C^∞ Urysohn lemma we can pick $\zeta_j \in C_c^\infty(U)$ such that $\zeta_j = 1$ on V_j. Given $\phi C_c^\infty(U)$, for j sufficiently large we have $\mathrm{supp}(\phi) \subset V_j$ and hence

$$\langle F, \phi \rangle = \langle F, \zeta_j \phi \rangle = \langle \zeta_j F, \phi \rangle.$$

Therefore $\zeta_j F \to F$ as $j \to \infty$.

Since supp(ϕ) is compact, $\zeta_j F$ can be regarded as a distribution on $\mathbb{R}^n$ Let ψ, ψ_t be as in Lemma 9.4. If $\tilde{\psi}(x) = psi(-x)$ then $\int \tilde{\psi} = 1$, so given $\phi \in C_c^\infty$ we have $\phi * \tilde{\psi}_t \to \phi$ in C_c^∞ by Lemma 9.4. By Proposition 9.3 we have $(\zeta_j F) * \psi^t \in C^\infty$ and

$$\langle (\zeta_j F) * \psi_t, \phi \rangle = \langle \zeta_j F, \phi * \tilde{\psi}_t \rangle \to \langle \zeta_j F, \phi \rangle,$$

so $(\zeta_j F) * \psi^t \to \zeta_j F$ in $\mathcal{D}'$. In other words, every neighborhood of F in $\mathcal{D}'(U)$ contains the C^∞ functions $(\zeta_j F) * \psi_t$ for j large and t small.

Finally, note that $\mathrm{supp}(\zeta_j) \subset V_k$ for some k. If $\mathrm{supp}(\phi) \cap \overline{V_k} = \emptyset$, then for small enough t we have $\mathrm{supp}(\phi * \tilde{\psi}) \cap \overline{V_k} = \emptyset$ (Lemma 9.4 again) and hence

$$\langle (\zeta_j F) * \psi_t, \phi \rangle = \langle F, \zeta_j (\phi * \tilde{\psi}_t) \rangle = .$$

In other words,

$$\mathrm{supp}((\zeta_j F) * \psi_t) \subset \overline{V_j} \subset U$$

so we are done. $\square$

Example: derivatives of step functions.

If $H = \chi_{(0,\infty)}$, then

$$\langle H', \phi \rangle = -\langle H, \phi' \rangle = -\int \mathbb{R} H \cdot \phi' dx = -\int_0^\infty H \cdot \phi' dx = \phi(0) \langle \delta, \phi \rangle$$

so $H' = \delta$ as distributions.

Example: divergent integral.

Let $f(x) = \chi_{(0,\infty)} \cdot \frac{1}{x}$.

This is locally integrable on $U = (0, \infty)$ so defines a distribution there. It does not define a distribution on $\mathbb{R}$ since $\int f\phi$ diverges unless $\phi(0) = 0$.

However, there is a distribution on $\mathbb{R}$ that agrees with f on U.

Note that $L(x)(\log x)\chi_{0,\infty)}$ defines a distribution on $\mathbb{R}$, so L' is a well defined distribution.

Let $L_\epsilon = (\log x)\chi_{(\epsilon,\infty)}$. By the dominated convergence theorem

$$\int L\phi = \lim_{\epsilon\to 0} \int L_\epsilon \phi$$

for $\phi \in \mathcal{S}$. Thus $L_\epsilon \to L$ as distributions and hence $L'_\epsilon \to L'$ as distributions. But

$$\langle L'_\epsilon, \phi\rangle = -\langle L_\epsilon, \phi'\rangle = -\int L_\epsilon \phi' = -\int_\epsilon^\infty \log(x)\phi'(x)dx = \int_\epsilon^\infty \frac{\phi(x)}{x}dx + \phi(\epsilon)\epsilon.$$

As $\epsilon \to 0$ this converges to $\langle L', \phi\rangle$ even though the two terms diverge.

Example: This function from calculus has different mixed partials.

$$f(x,y) = \frac{xy(x^2 - y^2)}{x^2 + y^2},$$

$$\partial_x \partial_y f(0,0) \neq \partial_y \partial_x f(0,0).$$

However mixed partials define the same function everywhere except at origin, so mixed partials in sense of distributions are the same. This is always true for distributions since it is true for C_c^∞.

Chapter 9.2: Compactly Supported, Tempered and Periodic Distributions

Distribution = $\mathcal{D}'$ = dual of $C_c^\infty(U)$.

Compactly supported distribution = $\mathcal{E}'$ = dual of $C^\infty(U)$.

Tempered distribution = $\mathcal{S}'$ = dual of $\mathcal{S}$, Schwarz class

Periodic distributions = $\mathcal{D}'(\mathbb{T}^n)$ = dual of $C^\infty(\mathbb{T}^n)$.

$C^\infty(U)$ is a Fréchet space with seminorms

$$\|f\|_{[m,\alpha]} = \sup_{V_m} |\partial^\alpha f(x))|,$$

where $\{V_m\}$ is an increasing sequence of precompact open subsets of U whose union is U.

Prop 9.7: $C_c^\infty(U)$ is dense in $C^\infty(U)$.

Proof. For each m use the C^∞ Urysohn lemma to pick $\psi_m \in C_c^\infty(U)$ with $\psi_m = 1$ on V_m. If $\phi \in C^\infty(U)$ then

$$\|\phi - \psi_k \phi\|_{[m,\alpha]} = 0$$

for $k \geq m$, so $\phi_k \phi \to \phi$ in $C^\infty(U)$. $\square$

Theorem 9.7: $\mathcal{E}'(U)$ is the dual space of $C\infty(U)$. More precisely: If $F \in \mathcal{E}'(U)$, then F extends uniquely to a continuous linear functional on $C^\infty(U)$, and if G is a continuous linear functional on $C^\infty(U)$, then G restricted to $C_c^\infty(U) \in \mathcal{E}'(U)$.

Proof. If $F \in E'(U)$, choose $\psi \in C_c^\infty(U)$ with $\psi = 1$ on $\mathrm{supp}(F)$, and define the linear functional G on $C^\infty(U)$ by

$$\langle G\phi \rangle = \langle F, \psi\phi \rangle.$$

Since F is continuous on $C_c^\infty(\mathrm{supp}(\psi))$, and the topology of the latter is defined by the norms $\phi \to \|\partial^\alpha \phi\|_u$, by Proposition 5.15 there are $N \in \mathbb{N}$ and $0 < C < \infty$ so that

$$|\langle G\phi \rangle| \leq C \sum_{|\alpha| \leq N} \sup_{x \in \mathrm{supp}(\psi}) |\partial^\alpha \phi(x)| \leq C' \sum_{|\alpha| \leq N} \|\phi\|_{[m,\alpha]}.$$

Thus G is continuous on $C^\infty(U)$. That G is the unique extension follows from the previous lemma ($C_c^\infty(U)$ is dense in $C^\infty(U)$).

On the other hand, if G is a continuous linear functional on $C^\infty(U)$, then by Proposition 5.15 there exist C, m, N such that

$$|\langle G\phi\rangle| \leq C \sum_{|\alpha|\leq N} \|\phi\|_{[m,\alpha]}.$$

Since $\|\phi\|_{[m,\alpha]} \leq \|\partial^\alpha \phi\|_u$, this implies G is continuous of $C_c^\infty(K)$ for each compact K in U. Thus G restricted to $C_c^\infty(U)$ is continuous and hence in $\mathcal{D}'(U)$.

Moreover, if $\mathrm{supp}(\phi)$ is disjoint from V_m then $\langle G, \phi\rangle = 0$ so $\mathrm{supp}(G)$ is in V_m and so G is compactly supported. $\square$

The operations of differentiation, multiplication by Coo functions, translation, and composition by linear maps discussed in 9.1 all preserve the class $\mathcal{E}'$.

Convolution, is slightly more complicated.

First, if $F \in \mathcal{E}'$ and $\phi \in C_c^\infty$ then $F * \phi \in C_c^\infty$ since Proposition 8.6d remains valid.

Second, if $f \in \mathcal{E}'$ and $\phi \in C^\infty$, then $F * \phi$ can be defined as a C^∞ function or as a distribution just as before:

$$F * \psi(x) = \langle F, \tau_n \psi \rangle,$$

$$\langle F * \psi, \phi(x) \rangle = \langle F, \phi * \tilde{\psi} \rangle,$$

for $\phi \in C_c^\infty$.

Finally, a further dualization allows us to define convolutions of arbitrary distributions with compactly supported distributions. If $F \in \mathcal{D}'$ and $G \in E'$, we can define $F * G \in \mathcal{D}'$ and $G * F \in \ D'$ as follows:

$$\langle F * G, \phi \rangle = \langle F, \tilde{G} * \phi \rangle$$

$$\langle F * G, \phi \rangle = \langle G, \tilde{F} * \phi \rangle$$

The proof that $F*G$ and $G*F$ are indeed distributions and that $F*G = G*F$ are Exercises 20 and 21.

We have not yet extended the Fourier transform to distributions.

Fact: If $\phi \in C_c^\infty$ is not zero then $\hat{\phi} \notin C_c^\infty$, in fact, its support is the entire space. Stronger versions of this are called the uncertainty principle.

Proof. Suppose $\hat{\phi}$ is zero on a neighborhood of ξ_0. By replacing ϕ by $]phie^{-2\pi i\xi\cdot x}$ we may assume $\xi_0 = 0$. Since ϕ has compact support we may expand $e^{-2\pi i\xi\cdot x}$ in a power series and integrate term-by-term

$$\begin{aligned}\hat{\phi}(\xi) &= \sum_{k=1}^{\infty} \frac{1}{k!} \int (-2\pi i\xi \cdot x)^k \phi(x) dx \\ &= \sum_{\alpha} \frac{1}{\alpha!} \xi^\alpha \int (-2\pi i x)^\alpha \phi(x) dx\end{aligned}$$

This uses the multinomial theorem (Exercise 8.2.a)

$$(x_1 + \cdots + x_n)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} x^\alpha.$$

But

$$\int (-2\pi x)^\alpha \phi(x)dx = \partial^\alpha \hat{\phi}(0) = 0$$

since $\hat{\phi}$ vanishes on a neighborhood of zero. Hence $\hat{\phi}$ is everywhere zero and so ϕ is too. $\square$

There are many more quantitative versions.

The Uncertainty Principle: A Mathematical Survey by Gerald B. Folland and Alladi Sitaram

Theorem: if $f \in \mathcal{S}$, $x_0, \xi_0 \in \mathbb{R}^n$ then

$$\|f\|_2 \leq 4\pi \{(x x 0) f(x)\|_2 \cdot \|(\xi - \xi_0) f(\xi)\|_2.$$

Theorem (Amrien and Berthier): if $f \in \mathcal{S}$, and $E, F \subset \mathbb{R}^n$ have finite measure then

$$\|f\|_{L^2(\mathbb{R}^n)} \leq C(E, F) \left(\|f\|_{L^2(\mathbb{R}^n \setminus E)} \|\hat{f}\|_{L^2(\mathbb{R}^n \setminus F)} \right)$$

Conclusion (Benedicks) : f and $\hat{f}$ can't both be supported on finite measure sets.

Hardy's Uncertainty Principle: Suppose f is a measurable function on $\mathbb{R}$ such that

$$|f(x)| \leq A e^{-\alpha \pi x^2}$$
$$|\hat{f}(x)| \leq B e^{-\beta \pi x^2}$$

If $\alpha\beta > 1$ then f is the zero function.

Theorem (Benedicts): If $f \in L^1(\mathbb{R}^n)$ let

$$A = \{x : |f(x)| > 0\}, \qquad = \{x : |\hat{f}(x)| > 0\}.$$

If $m(A) < \infty$ and $m(B) < \infty$ then $f = 0$ a.e.,

Proof. By dilating we may assume $m(A) < (2\pi)^n$ and $m(B) < \infty$. Let $\phi = \chi_B$ and

$$\tilde{\phi} = \sum_{k \in \mathbb{Z}^n} \tau_k \phi.$$

This is positive measurable and 1-periodic. Let $K = [0,1]^n$. Then

$$\int_K \tilde{\phi} = \int_{\mathbb{R}^n} \phi = m(B) < \infty$$

so for almost every ξ, we have $\xi + k \in B$ for only finitely many $k \in \mathbb{Z}^n$.

Fix $\xi_0 \in \mathbb{R}^n$ and define

$$\tilde{f}_{\xi_0}(x) = \sum_{k \in \mathbb{Z}^n} e^{-2\pi i \xi_0 \cdot (x-k)} f(x-k).$$

Then

(i) $\tilde{f}_{\xi_0} \in L^1(\mathbb{T}^n)$

(ii) $\tilde{f}_{\xi_0}$ has Fourier coefficients

$$(f_{\xi_0})^{\wedge}(k) = (2\pi)^{-n} \hat{f}(\xi_0 + k),$$

(iii) $m(\{x : \tilde{f}_{\xi_0}(x) > 0\}) < (2\pi)^n$

By (ii) and our earlier remarks, $\tilde{f}_{\xi_0}$ is a trigonometric polynomial for a.e. ξ_0. But by (iii) this trig polynomial vanishes on a set of positive measure, so is the zero function. Thus for a.e. ξ_0, $\hat{f}(\xi + k) = 0$ for all k. This implies $\hat{f} = 0$ a.e. and hence $f = 0$ a.e.. $\square$

Michael Benedicks, 1949–present

Even though the Fourier transform does not map C_c^∞ into itself, it does not $\mathcal{S}$ into itself.

Prop 9.9: Suppose $\psi \in C_c^\infty$, $\psi(0) = 1$ and let $\psi^\epsilon(x) = \psi(\epsilon x)$. Then for any $\phi \in \mathcal{S}$, $\psi^\epsilon \phi \to \phi$ in $\mathcal{S}$ as $\epsilon \to 0$. In particular, C_c^∞ is dense in $\mathcal{D}$.

Proof. First consider the semi-norms with no derivatives. Given any N and $\eta > 0$ we can choose a compact K so that

$$(1 + |x|)^N |\phi(x)| < \eta$$

off K. Since $\psi(\epsilon x) \to 1$ uniformly on K we get

$$\sup_K (1 + |x|)^N |\phi(x) - \psi^\epsilon(x)\phi(x)| \to 0$$

so $\|\phi - \psi^\epsilon \phi\|_N \to 0$.

For semi-norms involving derivatives, use the product rule:

$$(1+|x|)^N \partial^\alpha(\psi^\epsilon \phi - \phi) = (1+|x|)^N(\psi^\epsilon \partial^\alpha \phi - \partial^\alpha \phi) + E_\epsilon(x),$$

where E_ϵ is a sum of terms involving derivatives of ψ^ϵ. Since

$$\partial^\beta \psi^\epsilon(x)| = \epsilon^{|\beta|}|\partial^\beta \psi(\epsilon x)| \leq C_\beta \cdot \epsilon^{|\beta|},$$

we have $\|E_\epsilon\|_u \leq C\epsilon \to 0$. Thus $\|\psi^\epsilon \phi - \phi\|_{(N,\alpha)} \to 0$. $\square$

Defn: A **tempered distribution** is a continuous linear functional on $\mathcal{S}$. These are denoted $\mathcal{S}'$.

If $F \in \mathcal{S}'$ then it defines a distribution on C_c^∞ since convergence in C_c^∞ implies convergence in $\mathcal{S}$. Thus tempered distributions are a subset of distributions: the ones that extend continuously from C_c^∞ to $\mathcal{S}$.

Roughly speaking a distribution is tempered if it does not grow too quickly at ∞.

Compactly supported distributions are tempered.

If $f \in L^1_{\rm loc}$ and $\int (1+|x|)^N |f(x)|dx < \infty$ for some $N > -\infty$, then f is tempered.

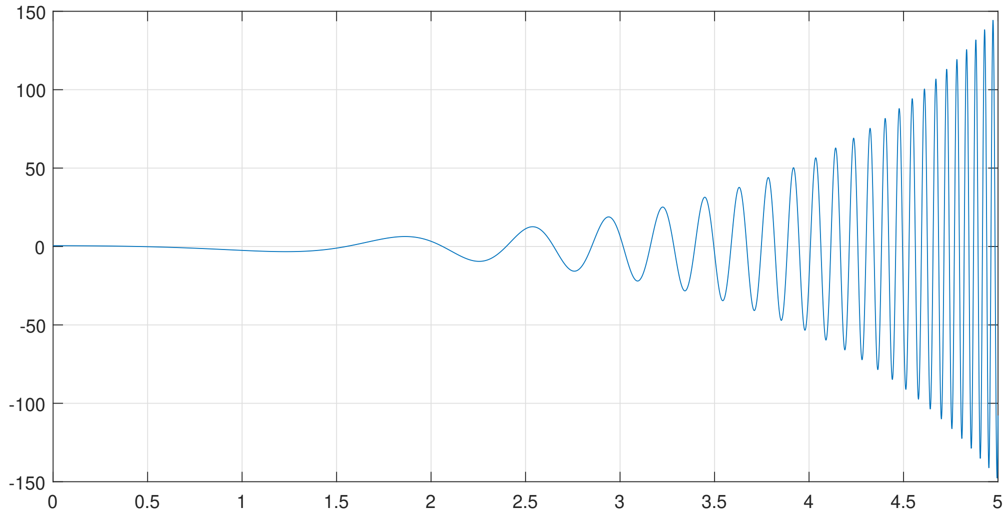
Example 1: e^{iax} is bounded, hence tempered by (ii).

Example 2: e^{bx} is not tempered: choose $\psi \in C_c^\infty$ so that $\int \psi = 1$. Then $\psi_j = e^{-bx}\psi(x-j) \to$ in $\mathcal{S}$, but

$$\int \phi_j e^{bx} dx = \int \psi dx = 1 \not\to 0.$$

Example 3: $f(x) = e^x \cos(e^x)$ is tempered, because it is the derivative of the bounded function $\sin(e^x)$. Indeed,

$$|\int f\phi| = |\int \phi'(x)\sin(e^x)dx| \le C\|\phi\|_{(2,1)}.$$



$$f(x) = e^x \cos(e^x)$$

Differentiation, translation and composition with linear transformations all work the same for tempered distributions as for distributions.

Multiplication by smooth functions is slightly different.

For $F \to \psi F$ to map a tempered distribution to a tempered distribution, we need ψ and its derivatives to have at most polynomial growth

$$|\partial^\alpha \psi(x)| \leq C^\alpha (1 + |x|)^{N(\alpha)}$$

Such functions are called **slowly increasing**.

Polynomials are examples. So is $(1 + |x|^2)^s$, $s \in \mathbb{R}$.

Prop. 9.10: If $F \in \mathcal{S}'$ and $\psi \in \mathcal{S}$ then $F * \psi$ is a slowly increasing C^∞ function, and for any $\phi \in \mathcal{S}$,

$$\int (F * \psi)\phi = \langle F, \phi * \tilde{\psi} \rangle.$$

Proof. That $F *' \psi \in C^\infty$ is proven as in Proposition 9.3.

By Proposition 5.15, the continuity of F implies that there exist m, N, C such that

$$|\langle F, \phi \rangle| \leq C \sum_{|\alpha| \leq N} \|\phi\|_{(,\alpha)},$$

and hence by (8.12),

$$\begin{aligned} |F * \psi| &\leq C \sum_{|\alpha| \leq N} (1 + |y|)^m |\partial^\alpha \psi(x - y)| \\ &\leq C(1 + |x|)^m \sum_{|\alpha| \leq N} \sup_y (1 + |x - y|)^m |\partial^\alpha \psi(x - y)| \\ &\leq C(1 + |x|)^m \sum_{|\alpha| \leq N} \|\psi\|_{(m\alpha)}. \end{aligned}$$

The same reasoning applies with ψ replaced by $\partial^\beta\psi$, so $F*\psi$ is slowly increasing.

Next, by Proposition 9.3 we know that the equation

$$\int (F*\psi)\phi = \langle F, \phi * \tilde{\phi}\rangle$$

holds when $\phi, \psi \in C_c^\infty$.

By Proposition 9.9, if $\phi, \psi \in \mathcal{S}$, we can find sequences $\{\phi_j\}$ and $\{\psi_j\}$ in C_c^∞ that converge to ϕ, ψ in $\mathcal{S}$. Then

$$\phi_j * \tilde{\psi}_j \to \phi * \tilde{\psi}$$

in $\mathcal{S}$ by the proof of Prop 8.11, so

$$\langle F, \phi_j * \tilde{\psi}_j\rangle \to \angle F, \phi * \tilde{\psi}\rangle.$$

On the other hand, the preceding estimates show that

$$|F * \psi_j(x)| \leq C(1 + |x|)^m,$$

with C and m independent of j, and likewise

$$|\phi_j(x)| \leq C(1 + |x|)^{-m-m-1}.$$

Thus

$$\int (F * \psi_j)\phi_j \to \int (F * \psi)\phi,$$

by the dominated convergence theorem. $\square$

The main reason for considering tempered distributions, is the Fourier transform.

Recall the Fourier transform maps $\mathcal{S}$ continuously into itself, and that for $f, g \in \mathcal{S} \subset L^1$,

$$\int \hat{f}(y)g(y)dy = \iint f(x)g(y)e^{-2\pi i x\cdot y}dxdy = \int f(x)\hat{g}(x)dx.$$

We can extend the Fourier transform to a continuous linear map from $\mathcal{S}'$ to itself by defining

$$\langle \hat{F}, \phi \rangle = \langle F, \hat{\phi} \rangle.$$

This definition agrees with the one in Chapter 8 when $f \in l^1 + L^2$. The basic properties of the Fourier transform in Theorem 8.22 continue to hold in $\mathcal{S}'$:

$$(\tau_y f)^\wedge = e^{-2\pi\xi\cdot y}\hat{F}, \qquad \tau_\eta \hat{F} = (e^{2\pi i\eta\cdot x}F)^\wedge,$$
$$\partial^\alpha \hat{F} = [(2\pi i x)^\alpha F]^\wedge \qquad (\partial^\alpha F)^\wedge = (2\pi i\xi)^\alpha \hat{F}$$

$$(f \circ T)^\wedge = |\det T|^{-1}\hat{f} \circ (T^*)^{-1}, T \in \mathrm{GL}(n, \mathbb{R}),$$

$$(F * \psi)^\wedge = \hat{\psi} \cdot \hat{F},$$

Verifications are left to the reader.

Inverse transform given by

$$\langle F^\vee, \phi \rangle = \langle F, \phi^\vee \rangle.$$

Fourier inversion:

$$\langle (\hat{F})^\vee, \phi \rangle = \langle \hat{F}, (\phi^\vee)^\wedge \rangle = \langle \hat{F}, \phi \rangle.$$

Thus the Fourier transform is an isomorphism on $\mathcal{S}'$.

There is an alternative way to define Fourier transform of a compactly supported distribution F. Since $\phi(x) = \exp(2\pi i\xi \cdot x)$ is C^∞, $\langle F, \phi\rangle$ should also be $\hat{F}$. The two possibilities agree:

Prop 9.11: If $F \in \mathcal{E}'$, then $\hat{F}$ is a slowly increasing C^∞ function, and it is given by

$$\hat{F}(\xi) = \langle F, E_{-\xi}\rangle, \text{ where } E_\xi = \exp(2\pi i\xi \cdot x).$$

Proof. Let $g(\xi) = \langle F, E_{-\xi}\rangle$. Consideration of difference quotients of g, as in the proof of Proposition 9.3, shows that $g \in C^\infty$ with derivatives given

$$\partial^\alpha g = \langle F, \partial_\xi^\alpha E_{-\xi}\rangle = (-2\pi i)^{|\alpha|}\langle F, x^\alpha E_{-\xi}\rangle.$$

Moreover, by Theorem 9.8 and Proposition 5.15, there exist m, N, C such that

$$\partial^\alpha g(\xi)| \leq C \sum_{|\beta|\leq N} \sup_{|x|\leq m} |\partial^\beta[x^\alpha E_{-\xi}(x)]| \leq C'(1+m)^{\alpha|}(1+|\xi|)^N,$$

so g is slowly increasing.

It remains to show that $g = \hat{F}$, and by Proposition 9.9 it suffices to show that

$$\int g\phi = \langle F, \hat{\phi} \rangle,$$

for any ϕC_c^∞. In this case $g\phi \in C_c^\infty$ so $\int g\phi$ can be approximated by Riemann sums as in the proof of Proposition 9.3, say

$$\int g\phi \approx \sum g(\xi_j))\phi(\xi_j)\Delta\xi_j.$$

The corresponding sums

$$\sum \phi(\xi_j) e^{-2\pi i \xi_j \cdot x}$$

and their derivatives in x converge uniformly, for x in any compact set, to $\phi(x)$ and its derivatives. Therefore, since F is a continuous functional on C^∞

$$\begin{aligned} \int g\phi &= \lim \sum \langle F, E_{-\xi} \rangle \phi(\xi_j) \Delta\xi_j \\ &= \lim \langle F, \sum \phi(x_j) E_{-\xi} \Delta\xi_j \rangle \\ &= \langle F, \hat{\phi} \rangle. \end{aligned}$$

□

The Fourier transform of the point mass at 0 is the constant function 1:

$$\langle \delta E_{-\xi} = E_{-\xi}(0) = 1.$$

More generally,

Prop. 9.12: The Fourier transform of the linear combinations of δ its derivatives are precisely the polynomials.

$$(x^\alpha)^\wedge = [(-x)^\alpha]^\vee = (2\pi i)^{-|\alpha|}\partial^\alpha \delta,$$
$$\hat{E}_y = (E - -y)^\vee = \tau_y \delta.$$

$$\int 1 \cdot e^{2\pi\xi\cdot x} d\xi = \delta(x).$$

Every distribution is a linear combination of derivatives of continuous functions.

Prop 9.14:

a. If $F \in \mathcal{E}'$, there exist $N \in \mathbb{N}$, constants C_α for $|\alpha| \leq n$ and $f_\alpha C_0(\mathbb{R})$ so that that $F = \sum_{|\alpha| \leq N} c_\alpha \partial^\alpha f$.

b. If $F \in \mathcal{D}'(U)$ and V is a precompact open set in U, then there are N, c_α, f_α as above so $F = \sum_{|\alpha| \leq N} c_\alpha \partial^\alpha f$ in V.

Proof. By Proposition 9.11, if $F \in \mathcal{E}'$ then $\hat{F}$ is slowly increasing, so

$$g(\xi) = (1 + |\xi|^2)^{-M} \hat{F}(\xi),$$

is in l^1 if M is large enough. Let $f = \hat{g}$. Then $f \in C_0$ and

$$\hat{F} = (1 + |\xi|^2)^M \hat{f}.$$

Thus

$$F = (I - \frac{1}{4\pi^2} \sum_1^n \partial_j^2)^M f.$$

This proves (a). For (b), choose $\psi \in C_c^\infty(U)$ such that $\psi = 1$ on V and apply (a) to ψF. $\square$

I am omitting Folland's remarks on periodic distributions. read about this in the textbook.

Chapter 9.3: Sobolev Spaces

Sobolev spaces measure smoothness properties of functions and distributions is in terms of L^2 norms.

- L^2 is Hilbert space,
- the Fourier transform converts differentiation into multiplication by the coordinate functions and, is an isometry on L^2.

Let H_k be the space of all functions $f \in L^2(\mathbb{R}^n)$ whose distribution derivatives ∂^α are L^2 functions for $|\alpha| \leq k$. Make this into a Hilbert space with the inner product

$$\langle f, g \rangle = \sum_{|\alpha| \leq k} (\partial^\alpha f)\overline{(\partial^\alpha g)}.$$

However, it is more convenient to use an equivalent inner product defined in terms of the Fourier transform. Theorem 8.22e and the Plancherel theorem imply that $f \in H_k$ iff

$$\xi^k \hat{f} \in L^2$$

for $|\alpha| \leq k$.

A simple modification of the argument in the proof of Proposition 8.3 shows that there exist $C_, C_2 > 0$ such that

$$C_1(1 + |\xi|^2)^k \leq \sum_{|\alpha| \leq k} |\xi^\alpha|^2 C_2(1 + |\xi|^2)^k.$$

If follows that $f \in H_k$ iff $(1 + |\xi|^2)^{k/2} \in L^2$. The norms

$$\left(\sum_{|\alpha| \leq k} \|\partial^\alpha f\|_2^2 \right)^{1/2}, \qquad \|(1 + |\xi|^2)^{k/2} \hat{f}\|_2$$

are equivalent.

The second norm makes sense for any $k \in \mathbb{R}$, and we can use it to extend the definition of H_s to all real s.

For any $s \in \mathbb{R}$,

$$\xi \to (1 + |\xi|^2)^{s/2}$$

is C^∞ and slowly increasing (Exercise 30), so

$$\Lambda_s f = [(1 + |\xi|^2)^{s/2} \hat{f}]^\vee$$

is a continuous linear operator on $\mathcal{S}'$. In fact, it is an isomorphism since $\Lambda_s^{-1} = \Lambda_) - s$.

Defn: If $s \in \mathbb{R}$ define the **Sobolev space** H_s to be

$$H_s = \{f \in \mathcal{S}' : \Lambda_s f \in L^2\},$$

with the inner product and norm

$$\langle f, g \rangle_{(s)} = \int (\Lambda_s f)\overline{(\Lambda_s g)} = \int \hat{f}(\xi)(1 + |\xi|^2)^s \overline{\hat{g}(\xi)}.$$

$$\|f\|_{(s)} = \|\Lambda_s f\|_2 = \left[\int \hat{f}(\xi)(1 + |\xi|^2)^s d\xi\right]^{1/2}.$$

(i) The Fourier transform in a unitary isomorphism from H_s to $L^2(\mathbb{R}^n, \mu_s)$ where $d\mu_s = (1 + |\xi|^2)^s d\xi$. So H_s is a Hilbert space.

(ii) $\mathcal{S}$ is dense in H_s for all $s \in \mathbb{R}$.

(iii) If $t < s$, H_s is dense in H_t in the topology of H_t and $\| \cdot \|_{(t)} \leq \| \cdot \|_{(t)}$.

(iv) Λ_t is a unitary isomorphism from H_s to H_{s-t} for all $s, t \in \mathbb{R}$.

(v) $H_0 = L^2$ and $\| \cdot \|_{(0)} \leq \| \cdot \|_{L^2}$.

(vi) ∂^α is a bounded linear map from H_s to $H_{s-|\alpha|}$ for all s, α because $|\xi^\alpha| \leq (1 + |\xi|^2)^{|\alpha|/2}$.

For $s \geq 0$ elements of H_s are functions. This need not be true for $s < 0$. For example the point mass $\delta \in H_s(\mathbb{R}^n)$ iff $s < -n/2$. (Recall $\hat{\delta}$ is the constant function 1.)

Prop 9.16: If $s \in \mathbb{R}$, the duality between $\mathcal{S}$ and $\mathcal{S}'$ induces a unitary isomorphism from H_{-s} to $(H_s)^*$. More precisely, if $f \in H_{-s}$ the functional

$$\phi \to \langle f, \phi \rangle,$$

on $\mathcal{S}$ extends to a continuous linear functional on H_s with operator norm equal to $\|f\|_{(s)}$ and every element of $(H_s)^*$ arises in this way.

Proof. If $f \in H_{-s}$ and $\phi \in \mathcal{S}$,

$$\langle f, \phi \rangle = \langle f^\vee, \hat{\phi} \rangle = \int f^\vee(\xi)\hat{\phi}(\xi)d\xi,$$

since $f^\vee(\xi) = \hat{f}(-\xi)$ is a tempered function. By the Schwarz inequality,

$$\begin{aligned} |\langle f, \phi \rangle| \;&\leq\; \left[\int |f^\vee(\xi)|^2(1+|\xi|^2)^{-s}d\xi\right]^{1/2} \left[\int |\hat{\phi}(\xi)|^2(1+|\xi|^2)^{s}d\xi\right]^{1/2} \\ &=\; \|f\|_{(-s)} \cdot \|\phi\|_{(s)}. \end{aligned}$$

Thus the functional $\phi \to \langle f, \phi \rangle$ extends continuously to H_s with norm at most $\|f\|_{(-s)}$.

In fact, the norm is equal to this since if $g \in \mathcal{S}'$ is the distribution whose Fourier transform equals

$$\hat{g}(\xi) = (1 + |\xi|^2)^{-s}\overline{\hat{f}(\xi)},$$

then $g \in H^s$ and

$$\langle f, g \rangle = \int |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi = \|f\|^2_{(-s)} = \|f\|_{(-s)} \|f\|_{(s)}.$$

Finally, if $G \in (H_s)^*$ then $G \circ \mathcal{F}^{-1}$ is a bounded linear functional on $L^2(\mu_s)$ where $d\mu_s = (1 + |\xi|^2)^s d\xi$, so there is a $g \in L^2(\mu_s)$ so that

$$G(\phi) = \int \hat{\phi}(\xi) g(\xi) (1 + \xi|^2)^2 d\xi.$$

But then $G(\phi) = \langle f,]phi \rangle$ where

$$f^\vee(\xi) = (1 + |\xi|^2)^s g(\xi),$$

and $f \in H_{-s}$ since

$$\|f\|^2_{(-s)} = \int |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi = \int |g(\xi)|^2 (1 + |\xi|^2)^s d\xi.$$

$\square$

Elements of H_s that are functions are define a.e.

To ask if such a function is C_k means that it agrees a.e. with a C^k function.

Define

$$D_0^k = \{f \in C^k(\mathbb{R}^n) : \partial^\alpha f \in C_0 \text{ for } |\alpha| \leq k\}.$$

This is a Banach space with the norm

$$\sum_{|\alpha|\leq k} \|\partial^\alpha f\|_u.$$

The Sobolev Embedding Theorem: Suppose $s > k + \frac{1}{2}n$.

(a) If $f \in H_s$, then $(\partial^\alpha f)^\wedge \in L^1$ and $\|(\partial^\alpha f)^\wedge\|_1 \leq C\|f\|_{(s)}$, where C depends only n $k - s$.

(b) $H_s \subset C_0^k$ and the inclusion map is continuous.

Proof. By the Schwarz inequality

$$\begin{aligned}
(2\pi)^{|\alpha|}\int |(\partial^\alpha f)^\wedge| d\xi &= \int |\xi^\alpha \hat{f}(\xi)| d\xi \\
&\leq \int |(1+|\xi|^2)^{k/2}\hat{f}(\xi)| d\xi \\
&\leq \left[\int |(1+|\xi|^2)^s \hat{f}(\xi)|^2 d\xi\right]^{1/2}\left[\int |(1+|\xi|^2)^{k-s} d\xi\right]^{1/2} \\
&\leq \|f\|_{(s)}\cdot C.
\end{aligned}$$

This proves (a). Part (b) follows from the Fourier inversion formula and the Riemann-Lebesgue lemma. $\square$

Cor 9.18: If $f \in H_s$ for all s, then $f \in C^\infty$.

Example: Let $f_\lambda(x) = x^\lambda \phi(x)$ where $\phi \in C_c^\infty$ and ϕ equals 1 on a neighborhood of 0. Then $\partial^\alpha f$ is C^∞ except at 0 and

$$|\partial^\alpha f_\lambda| \leq C_{\alpha,\lambda}|x|^{\lambda-|\alpha|}.$$

This is in L^1 iff $\lambda - |\alpha| > -n$ in which case the pointwise derivative $\partial^\alpha f$ is also the distributional derivative.

Moreover, $\partial^\alpha f_\lambda \in L^2$ iff $\lambda - |\alpha| > -n/2$ so $f \in H_k$ iff $\lambda > k - n/2$ whereas $f_\lambda \in C_0^k$ iff $\lambda > k$.

Lemma 9.19: For all $\xi, \eta \in \mathbb{R}^n$ and $s \in \mathbb{R}$,

$$(1+|\xi|^2)^s(1+|\eta|^2)^{-s} \leq 2^{|s|}(1+|\xi-\eta|^2)^s(1+|\eta|^2)^s.$$

Proof. Since $|\xi| \leq |\xi-\eta| + |\eta|$, we have

$$|\xi|^2 \leq 2(|\xi-\eta|^2 + |\eta|^2),$$

and

$$1+|\xi|^2 \leq 2(1+|\xi-\eta|^2)(1+|\eta|^2).$$

If $s \geq 0$ raise both sides to the power$s$. If $s < 0$, interchange η and ξ and replace s by $-s$ to get

$$(1+|\eta|^2)^{-s} \leq 2^{-s}(1+|\xi|^2)^{-s}(1+|\xi-\eta|^2)^{-s}.$$

$\square$

Theorem 9.20: Suppose that $\phi \in C_0(\mathbb{R}^n)$ and that $\hat{\phi}$ is a function that satisfies

$$\int (1+|\xi|^2)^{a/2}|\hat{\phi}(\xi)|d\xi = C < \infty,$$

for some $a > 0$. The the map $M_\phi(f) = \phi f$ is a bounded operator on H_s for $|s| \leq a$.

Proof. Since Λ_s is a unitary map from H_s to $H_0 = L^2$, it suffices to show that $\Lambda_s M_\phi \Lambda_{-s}$ is a bounded operator on L^2. But

$$(\Lambda_s M_\phi \Lambda_{-s} f)^\wedge(\xi) = (1+|\xi|^2)^{s/2}[\hat{\phi} * (\Lambda_{-s}f)^\wedge](\xi) = \int K(\xi,\eta)\hat{f}(\eta)d\eta,$$

where

$$K(\xi,\eta) = 1+|\xi|^2)^{s/2}(1+|\eta|^2)^{-s/2}\hat{\phi}(\xi-\eta).$$

By Lemma 9.19

$$|K(\xi,\eta)| \leq 2^{|s|/2}(1+|\xi-\eta|^2)^{|s|/2}|\hat{\phi}(\xi-\eta)|$$

so if $|s| \leq a$ then $\int |K(\xi,\eta|d\xi$ and $\int |K(\xi,\eta|d\xi$ are bounded by $2^{a/2}C$. Boundedness of $\Lambda_s M_\phi \Lambda_{-s}$ follows from the Plancherel theorem and Theorem 6.18. $\square$

Theorem 6.18: Suppose

$$\int K(x,y)d\mu(x) \le C, \qquad \int K(y,x)d\nu(x) \le C$$

for a.e. x and y. If $1 \le p \le \infty$ and $f \in L^p(d\nu)$ then

$$Tf(x) = \int K(x,y)f(y)d\nu(y)$$

converges absolutely for a.e. x and $Tf \in L^p(d\mu)$ with $\|Tf\|_p \le C\|f\|_p$.

Cor 6.18: If $\phi \in \mathcal{S}$ then M_ϕ is a bounded operator on every H_s, $s \in \mathbb{R}$.

Rellich's Theorem: Suppose that $\{f_k\}$ is a sequence of distributions in H_s, that are all supported in a fixed compact set K and satisfy $\sup_k \|f_k\|_{(s)} < \infty$. Then there is a subsequence $\{f_{k_j}\}$ that converges in H_t, for all $t < s$.

Proof. First we observe that by Proposition 9.11, $\hat{f}_k$ is a slowly increasing C^∞ function. Pick $\phi \in C_c^\infty$ such that $\phi = 1$ on a neighborhood of K. Then $f_k = \phi f_k$, so $\hat{f}_k = \hat{\phi} * \hat{f}_k$ where the convolution is defined pointwise by an absolutely convergent integral. By Lemma 9.19 and the Schwarz inequality,

$$\begin{aligned}(1+|\xi|^2)^{s/2}|\hat{f}_k(\xi)| &\leq 2^{|s|/2}\int |\hat{\phi}(\xi-\eta)|(1+|\xi-\eta|^2)^{|s|/2}|\hat{f}_k(\eta)|(1+|\eta|^2)^{s/2}d\eta \\ &\leq 2^{|s|/2}\|\phi\|_{(s)}\|f_k\|_{(s)} \\ \leq C < \infty\end{aligned}$$

Likewise, since

$$\partial_j(\hat{\phi} * \hat{f}_k) = (\partial_j \hat{\phi}) * \hat{f}_k$$

we see that $(1 + |\xi|^2)^{s/2} |\partial_j \hat{f}_k(\xi)|$ is bounded by a constant independent of ξ, j and k. In particular, the f_k's and their first derivatives are uniformly bounded on compact sets, so by the mean value theorem and the Arzelá-Ascoli theorem there is a subsequence $\{\hat{f}_{k_j}\}$ that converges uniformly on compact sets.

We claim that this subsequence is Cauchy in H_t for $t < s$. For any $R > 0$ we can write

$$\|f_{k_i} - f_{k_j}\|^2_{(s)} = \int (1 + |\xi|^2)^t |\hat{f}_{k_i} - \hat{f}_{k_j}|^2 d\xi,$$

as the sum of integrals over the regions $\{|\xi| \leq R\}$ and $\{|\xi| > R\}$. On the first region we have

$$(1 + |\xi|^2)^t \leq (1 + R^2)^{\max(t,0)}$$

and for $|\xi| > R$ we use

$$(1 + |\xi|^2)^t \leq (1 + R^2)^{t-s}(1 + |\xi|^2)^s.$$

These give

$$\begin{aligned} \|f_{k_i} - f_{k_j}\|^2_{(t)} \;\leq\; & ct^N (1 + R^2)^{\max(t,0)} \sup_{|xi| \leq R} |\hat{f}_{k_i} - \hat{f}_{k_j}|^2(\xi) \\ & + (1 + R^2)^{t-s} \|f_{k_i} - f_{k_j}\|^2_{(s)} \end{aligned}$$

For any $\epsilon > 0$ the second term will be less than $\epsilon/2$ if R is large enough since $t - s < 0$. Fixing such an R we then take i, j sufficiently large to make the first term $< \epsilon/2$. $\square$

Sobolev spaces can also be defined on proper open subsets $U \subset \mathbb{R}^n$. The **localized Sobolev space** $H_s^{\text{loc}}(U)$ is the set of all distributions $f \in \mathcal{D}'(U)$ such that for every precompact open set $V \subset U$ there exists $g \in H_s$ so that $f = g$ on V.

Prop 9.23: A distribution $f \in \mathcal{D}'(U)$ is in $H_s^{\text{loc}}(U)$ iff $\phi f \in H_s$, for every $\phi \in C_c^\infty(U)$.

Proof. If $f \in H_s^{\text{loc}}(U)$ and $\phi \in C_c^\infty(U)$, then f agrees with some $g \in H_s$ on a neighborhood of $\text{supp}(\phi)$; hence $\phi f = \phi g \in H_s$ by Corollary 9.21.

For the converse, given a precompact open $\subset U$, we can choose $\phi C_c^\infty(U)$ with $\phi = 1$ on a neighborhood of V by the C^∞ Urysohn lemma. Then $\phi f \imath H_s$ and $\phi f = f$ on V. $\square$

Application of Sobolev spaces to PDE:

Consider a constant-coefficient differential operator

$$P(D) = \sum_{|\alpha| \leq m} c_\alpha D^\alpha.$$

Assume $c_\alpha \neq 0$ for some $|\alpha| = m$.

The **principle symbol** P_m is the polynomial corresponding to the degree m terms of $P(D)$:

$$P_m(\xi) = \sum_{|\alpha| = m} c_\alpha \xi^\alpha.$$

Assume $c_\alpha \neq 0$ for some $|\alpha| = m$.

$P(D)$ is called **elliptic** if $P_m(\xi) \neq 0$ for all non-zero $\xi \in \mathbb{R}^n$. The Laplacian Δ is elliptic on $\mathbb{R}^n$ but the heat operator $\partial_t - \Delta$ and wave operator $\partial_t^2 - \Delta$ are not.

Lemma 9.24: Suppose $P(D)$ is of order m.T hen $P(D)$ is elliptic iff there are $0 < C, R < \infty$ so that $|P(\xi)| \geq C|\xi|^m$ when $|\xi| \geq R$.

Proof. If $P(D)$ is elliptic let $C_1 > 0$ be the minimum value of the symbol P_m on the unit sphere $|\xi| = 1$. Since P_m is homogeneous of degree m this implies $|P_m(\xi)| \geq C_1|\xi|^m$ for all ξ.

Moreover, $P - P_m$ is of order $m - 1$, so

$$|P(\xi) - P_m(\xi)| \leq C_2|\xi|^{m-1},$$

for some $C_2 < \infty$. Thus

$$|P(\xi)| \geq |P_m(\xi)| - |P(\xi) - P_m(\xi)| \geq \frac{1}{2}C_2|\xi|^m,$$

for $|\xi| \geq 2C_2/C_1$.

Conversely, if $P(D)$ is not elliptic, say $P(\xi_0) = 0$, then $|P(\xi)| \leq C|\xi|^{m-1}$ for all scalar multiples of ξ_0. $\square$

Lemma 9.25: If $P(D)$ is elliptic of order m, $u \in H_s$ and $P(D)u \in H_2$, then $u \in H_{s+m}$.

Proof. By hypothesis

$$(1 + |\xi|^2)^{s/2} \hat{u} \in L^2, \qquad (1 + |\xi|^2)^{s/2} P\hat{u} \in L^2.$$

By the previous lemma (Lemma 9.24) for some $R \geq 1$,

$$(1 + |\xi|^2)^{m/2} \leq 2^m |\xi|^m \leq c^{-1} 2^m |P(\xi)|$$

for $|\xi| \geq R$ and

$$(1 + |\xi|^2)^{m/2} \leq (1 + R^2)^{m/2}$$

for $|\xi| \leq R$. Thus

$$(1 + |\xi|^2)^{(s+m)/2} |\hat{u}| \leq C'(1 + |\xi|^2)^{s/2} (|P\hat{u}| + |\hat{u}|) \in L^2.$$

Hence $u \in H_{s+m}$. $\square$

9.26 The Elliptic Regularity Theorem: Suppose L is a constant coefficient elliptic differential operator of order m, $\Omega \subset \mathbb{R}^n$ is open, and $u \in \mathcal{D}'(\Omega)$. If $Lu \in H_s^{\mathrm{loc}}(\Omega)$ for some $s \in R$, then $u \in H_{s+m}^{\mathrm{loc}}(\Omega)$. If Lu is C^∞ on Ω, so is u.

Proof. We only need to prove the first claim.

By Prop 9.23 it suffices to show that $u\phi \in H_{s+m}$ for every $\phi \in C_c^\infty(\Omega)$. Let $V \subset U$ be a precompact open set containing $\mathrm{supp}(\phi)$ and choose $\psi \in C_c^\infty(\Omega)$ with $\psi = 1$ on $\overline{V}$.

Then $\psi u \in \mathcal{E}'$ so by Prop 9.11 it follows that $\psi u \in H_\sigma$ for some σ. By decreasing σ we may assume $s + m - \sigma$ is a positive integer.

Set $\psi_0 = \psi$ and $\psi_k = \phi$ and recursively choose $\psi_1, \ldots, \psi_{k-1} \in C_c^\infty$ so that $\psi_j = 1$ on a neighborhood of $\mathrm{supp}(\phi)$ and so that $\mathrm{supp}(\psi_j)$ is contained in the set where $\psi_{j-1} = 1$.

We claim that $\psi_j u \in H_{\sigma+j}$. When $j = k$ this gives

$$\phi u = \psi_k u = \in H_{\sigma+k} = H_{s+m}.$$

Thus it suffices to prove the claim by induction on j.

Note that for any $\zeta \in C_c^\infty$, the operator (a commutator)

$$[L, \zeta]f = L(\zeta f) - \zeta L f,$$

is a differential operator of order $m - 1$; by the product rule, the order m derivatives of f cancel out. The coefficients of $[L, \zeta]$ are linear combinations of derivatives of ζ and hence they vanish where ζ is constant.

Thus if $f \in H_t$ and $|\alpha| \leq m - 1$, we have $\partial^\alpha f \in H_{t-(m-1)}$ and thus $[L, \zeta]f \in H_{t-(m-1)}$ by Theorem 9.20 (multiplication by functions in C_c^∞ is a bounded operator).

To begin the induction, note that for $j = 0$ we have $\psi_0 u \in H_\sigma$ by our choice of σ.

In general, assume $\psi_j u \in H_{\sigma+j}$. Then since

$$\psi_{j+1} u == \psi_{j+1} \psi_j u$$

and

$$s = \sigma + k - m \geq \sigma + (j+1) - m$$

we have

$$\begin{aligned} L(\psi_{j+1} u) &= \psi_{j+1} L u + [L, \psi_{j+1}] u \\ &= \psi_{j+1} L u + [L, \psi_{j+1}] \psi_j u \\ &\in H_s + H_{\sigma+j-(m-1)} \\ &= H_{\sigma+j+1-m)}. \end{aligned}$$

Lemma 9.25 with $P(D) = L$ implies $\psi_{j+1} u \in H_{\sigma+j+1}$. $\square$

Example (Weyl's lemma) : Every distributional solution of $\Delta u = 0$ is C^∞ Thus harmonic functions are C^∞. So are solutions of $\Delta u = \phi$ where ϕ is C^∞.

Example (Cauchy-Riemann: If $L = \partial_1 + i\partial_2$ on $\mathbb{R}^2$ then

$$P(L) = \xi_1 + i\xi_2,$$

$$|P(\xi)| \geq (|\xi_1|^2 + |\xi_2|^2)^{1/2} = |\xi|,$$

Thus every distributional solution of $Lu = 0$ is C^∞. Thus holomorphic functions are C^∞.