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REAL ANALYSIS II

FOLLAND'S REAL ANALYSIS: CHAPTER 7
RADON MEASURES

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1. Chapter 7: Radon Measures

Chapter 7: Radon Measures

7.1 Positive linear functionals on $C_c(X)$

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Chapter 7.1: Positive linear functionals

X = locally compact Hausdorff space (LCH space) .

$C_c(X)$ = continuous functionals with compact support.

Defn: A linear functional I on $C_0(X)$ is positive if $I(f) \geq 0$ whenever $f \geq 0$,

Example: $I(f) = f(x_0)$ (point evaluation)

Example: $I(f) = \int f d\mu$, where μ gives every compact set finite measure.

We will show these are only examples.

Prop. 7.1; If I is a positive linear functional on $C_c(X)$, for each compact $K \subset X$ there is a constant C_K such that

$$|I(f)| \leq C_L \|f\|_u$$

for all $f \in C_c(X)$ such that $\text{supp}(f) \subset K$.

Proof. It suffices to consider real-valued I. Given a compact K, choose $\phi \in C_c(X, [0,1])$ such that $\phi = 1$ on K (Urysohn's lemma). Then if $\text{supp}(f) \subset K$,

$$|f| \leq \|f\|_u \phi,$$

or

$$\|f\|\phi - f > 0, , \quad \|f\|\phi + f > 0,$$

so

$$\|f\|_u I(\phi) - I)f) \geq 0, \quad \|f\|_u I(\phi) + I)f) \geq 0.$$

Thus

$$|I(f)| \leq I(\phi)\|f\|_u.$$

$\square$

Defn: let μ be a Borel measure on X and E a Borel subset of X. μ is called outer regular on E if

$$\mu(E) = \inf\{\mu(U) : U \supset E, U \text{ open }\},$$

and is inner regular on E if

$$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ open }\}.$$

Defn: if μ is outer and inner regular on all Borel sets, then it is called regular.

It turns out that regularity is a bit too much to ask for when X is not a-compact, so we adopt the following definition.

Defn: A Radon measure on X is a Borel measure that is finite on all compact sets, outer regular on all Borel sets, and inner regular on all open sets.

Later we prove that that Radon measures are also inner regular on all of their σ-finite sets.

Notation: we write $f \prec U$ if f U is open in X, $0 \leq f \leq 1$ and $\text{supp}(f) \subset U$.

Johann Radon (1887–1956)

The Riesz Representation Theorem: If I is a positive linear functional on $C_c(X)$, then there is a unique Radon measure μ on X so that $I(f) = \int f d\mu$ for all $f \in C_c(X)$. Moreover,

$$\mu(U) = \sup\{I(f) : f \in C_c(X), f \prec U\} \tag{7.3}$$

for all open sets U and

$$\mu(K) = \int \{I(f) : f \in C_c(X), f \geq \chi_K\} \tag{7.4}$$

for all compact sets K in X.

Frigyes Riesz (1880–1956)

Marcel Riesz (1886–1969)

Shizuo Kakutani (1911–2004)

Andrei Andreyevich Markov (1856–1922)

Proof of Riesz Representation:

Uniqueness: If $I(f) = \int f d\mu$ then $I(f) \le \mu(U)$ whenever U is open and $f \prec U$.

If $K \subset U$ is compact, then by Urysohn's lemma there is a $f \in C_c(X)$ so that $f = 1$ on K and $f \prec U$, so $\mu(K) \le \inf f d\mu \le \mu(U)$. Since μ is inner regular, we can choose $\mu(K) \nearrow \mu(U)$ so (7.3) holds. This determines μ on open sets, and hence on all Borel sets by outer regularity.

Existence: Define

$$\mu(U) = sup\{I(f) : f \in Cc(X), f \prec U\}.$$

for U open. Define

$$\mu^*(E) = \inf\{\mu(U) : U \supset E, U \text{ open }\},$$

for general sets E. Clearly $\mu(U) \leq \mu(V)$ if $U \subset V$, so $\mu^*(U) = \mu(U)$ if U is open.

Step 1: μ^* is an outer measure.

Step 2: Every open set is μ^*-measurable.

This implies every Borel set is μ^*-measurable, so restricting it to Borel sets gives a Borel measure satisfying (7.3) by definition.

Step 3: μ satisfies (7.4)

Thus μ is finite on compact sets.

Lemma: μ is inner regular on open sets.

Proof. If U is open and $\mu(U) > \alpha$ choose $f \in C_c(X)$ so that $f \prec U$ and $I(f) > \alpha$. Let $K = \text{supp}(f)$. If $g \in C_c(X)$and $g \geq \chi_K$ then $g - f \geq 0$ so $I(g) \geq I(f) > 0$. But then $\mu(K) > \alpha$ by (7.4), so μ is inner regular on U. $\square$

Step 4: $I(f) = \int f d\mu$ for all $f \in C_c(X)$.

Proof of Step 1:

By Proposition 1.10, if is enough to show that given open sets $\{U_j\}$ and $U = \cup_j U_j$ we have

$$\mu(U) \leq \sum \mu(U_j).$$

Choose $f \in C_c(X)$ with $f \prec U$ and set $K = \text{supp}(f)$. Then K is in a finite union of the U_j, so my Prop 4.41 there are $g_j \in C_c(X)$ with $g_j \prec U_j$ and $\sum g_k = 1$ on K. Then $f = \sum g_j f$ and $f g_j \prec U_j$ so

$$I(f) - \sum_1^n I(f g_j) \leq \sum_1^n \mu(U_j) \leq \sum_1^\infty \mu(U_j).$$

Taking the supremum over $f \prec U$ and using definition of $\mu(U)$, we get

$$\mu(U) \leq \sum_1^\infty \mu(U_j).$$

Proof of Step 2: Suppose U is open and E is any subset of X with finite μ^* measure. We must show

$$\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \setminus U).$$

First suppose E is open. Then $E \cap U$ is open, so given $\epsilon > 0$ we can find $f \in C_c(X)$ such that $f \prec E \cap U$ and $I(f) > \mu(EnU) - \epsilon$. Also, $E \setminus \mathrm{supp}(f)$ is open, so we can find $g \in C_c(X)$ such that $g \prec E \setminus \mathrm{supp}(f)$ and $I(g) > \mu(E \setminus \mathrm{supp}(f)) - \epsilon$. But then $f + f \prec E$ so

$$\begin{aligned} \mu(E) &\geq I(f) + I(g) \\ &\geq \mu(E \cap U) + \mu(E \setminus \mathrm{supp}(f)) - 2\epsilon \\ &\geq \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\epsilon \end{aligned}$$

Taking $\epsilon \to 0$ gives the desired estimate.

For general E, if $\mu^*(E) < \infty$, then there is an open V containing E so that $\mu(V) \leq \mu^*(E) + \epsilon$ and hence

$$\begin{aligned} \mu^*(E) + \epsilon &\geq \mu(V) \\ &\geq \mu(V \cap U) + \mu(V \setminus \operatorname{supp}(f)) \\ &\geq \mu^*(E \cap U) + \mu^*(E \setminus U) \end{aligned}$$

so taking $\epsilon \to 0$ finishes Step 2.

Proof of Step 3: If K is compact, $f \in C_c(X)$, and $f \geq \chi_K$, define

$$U_\epsilon = \{x : f(x) > 1 - \epsilon\}.$$

Then U_ϵ is open, and if $g \prec U_\epsilon$, we have $f/(1-\epsilon) - g \geq 0$ and so $I(g) \leq I(f)/(1-\epsilon)$. Thus

$$\mu(K) \leq \mu(U_\epsilon) \leq I(f)/(1-\epsilon)$$

and taking $\epsilon \to 0$ gives

$$\mu(K) \leq I(f)\mu(U).$$

On the other hand, for any open $U \supset K$, by Urysohn's lemma there exists $f \in C_c(X)$ such that $f \geq \chi_K$ and $f \prec U$, whence $I(f) \leq \mu(K)$. Since μ is outer regular on K , (7.4) follows.

Proof of Step 4: If suffices to show that $I(f) = \inf t f d\mu$ if f takes values in $[0,1]$ since $C_c(X)$ is the linear span of such functions.

Given a natural number N, let $K_j = \{x : f(x) \geq j/N\}$ and let $K_0 = \mathrm{supp}(f)$. Also, define $f_1, \ldots, f_N$ in $C_c(X)$ by

$$f_j(x) = \begin{cases} 0, & x \notin K_j \\ f(x) - (j-1)/N, & x \in K_{j-1} \setminus K_j \\ 1/N, & x \in K_j \end{cases}$$

Then $\chi_{K_j}/N \leq f_j \leq \chi_{K_{j-1}}/N$ and so

$$\frac{1}{N}\mu(K_j) \leq \int f_j d\mu \leq \frac{1}{N}\mu(K_j).$$

If U is open an contains K_j then $N f_j \prec U$ so $I(f_j) \leq mu(U)/N$.

By (7.4) $\frac{1}{N}\mu(K_j) \leq I(f_j)$ and by outer regularity, $I(f_j) \leq \frac{1}{N}\mu(K_j)$.

Set $f = \sum_1^N f_j$. Then

$$\frac{1}{N}\sum_1^N \mu(K_j) \leq \sum_1^N f_j d\mu = \int f d\mu \leq \frac{1}{N}\sum_0^{N-1} \mu(K_j),$$

$$\frac{1}{N}\sum_1^N \mu(K_j) \leq I(f) \leq \frac{1}{N}\sum_0^{N-1} \mu(K_j),$$

hence

$$|I(f) - \int f d\mu| \leq \frac{1}{N}(\mu(K_0) - \mu(K_N)) \leq \frac{\mu(\mathrm{supp}(f))}{N}.$$

Since $\mu(\mathrm{supp}(f)) < \infty$, taking $N \nearrow \infty$, proves $I(f) = \int f d\mu$. $\square$

Example: Harmonic measure

Suppose Ω is a bounded domain in ^{n} and for every continuous f on $\partial\Omega$ there is a harmonic function u on Ω that has a continuous extension to $\partial\Omega$ that equals f.

u is called the solution to the Dirichlet problem with boundary value f.

For $z \in \Omega$ the map $f \to u(z)$ is a positive linear functional.

By the Riesz theorem, there is a measure ω_z on $\partial\Omega$ so that

$$u(z) = \int_{\partial\Omega} f(x) d\omega_z(x),$$

ω_z is called harmonic measure with basepoint z.

For $E \subset \partial\Omega$, $\omega(z, E, \Omega)$ denotes harmonic measure of E in Ω with respect to z. It is a harmonic function in z with boundary values 1 on E and zero off E.

On the disk, harmonic measure equals Poisson kernels.

In general, harmonic measure equals first hitting distribution on $\partial\Omega$ of Brownian motion started at z.

Geometric properties of ω are widely studied.

F. and M. Riesz theorem: if if $\Omega \subset \mathbb{R}^2$ is simply connected, and $\partial\Omega$ is a rectifiable curve, then harmonic measure is absolutely continuous to length measure (same null sets).

Makarov's theorem: if $\Omega \subset \mathbb{R}^2$ is simply connected, then there is a set of Hausdorff dimension 1 on the boundary that has full harmonic measure. Every boundary set of Hausdorff dimension < 1 has zero harmonic measure.

In this case, harmonic measure is image of Lebesgue measure on circle under Riemann mapping.

Obvious generalization to $\mathbb{R}^n$ is wrong (Wolff snowflakes). Some partial results, but correct conjecture still unknown.

Bishop-Jones theorem: if $\Omega \subset \mathbb{R}^2$ is simply connected, and $E \subset \partial\Omega$ lies on some rectifiable curve, then $\ell(E) = 0$ implies $\omega(E) = 0$.

Versions of this recently proven in higher dimensions.

C. Bishop PhD thesis: if $\partial\Omega$ is a "fractal" curve then harmonic measure for inside and outside domains are singular (iff set of tangent points has zero linear measure).

Chapter 7.2: Regularity and approximation theorems

Prop. 7.5: Every Radon measure is inner regular on all of its σ-finite sets.

Proof. Suppose that μ is Radon and E is σ-finite (a countable union of finite measure sets). If $\mu(E)) < \infty$, the for any $\epsilon > 0$ we can choose an open U containing E such that $\mu(U) < \mu(E) + \epsilon$ and a compact $F \subset U$ such that $\mu(F) > \mu(U) - \epsilon$.

Since $\mu(U \setminus E) < \epsilon$, we can also choose an open $V \supset U \setminus E$ such that $\mu(V) < \epsilon$. Let $K = F \setminus V$. Then K is compact, is inside E,

$$\mu(K) = \mu(F) - \mu(F \cap V) > \mu(E) - \epsilon - \mu(V) > \mu(E) - 2\epsilon.$$

Thus μ is inner regular on E (if E has finite measure).

If $\mu(E) = \infty$, the $E = \cup E_j$ is an increasing union of finite measure sets. Thus for any $N \in \mathbb{N}$ there exists j such that $\mu(E_j) > N$ and hence, by the preceding argument, a compact $K \subset E_j$ with $\mu(K) > N$. Hence μ is inner regular on E in general. $\square$

Cor. 7.6 Every σ-finite Radon measure is regular. If X is σ-compact, every Radon measure on X is regular.

Proof. By definition, a Radon measure is outer regular, so inner regular implies regular.

A Radon measure gives finite mass to compact sets, so a Radon measure on a σ-compact space is automatically σ-finite. $\square$

Prop. 7.7: Suppose that μ is a σ-finite Radon measure on X and E is a Borel set in X.

a. For every $\epsilon > 0$ there exists an open U and a closed $F \subset E \subset U$ $\mu(U\ F) < \epsilon$.

b. There exists an F_σ set A and a G_δ set B such that $A \subset E \subset B$ and $\mu(B\mathcal{A}) = O$.

Defn: An F_σ **set** is a countable union of closed sets. A G_delta **set** is a countable union of open sets.

These are the first steps on the Borel hierarchy of sets.

Félix Edouard Justin Émile Borel (1871–1956)

Proof of Prop 7.7. Proof. Write $E = \cup E_j$ where the E_j's are disjoint and have finite measure. For each j, choose an open $U_j \supset E_j$ with $\mu(U_j)) < \mu(F_) + \epsilon 2^{-j-1}$ and let $U = \cup U_j$. This is open, contains E and $\mu(V \setminus E^c) = \mu(V \cap E) < \epsilon/2$.

By the same argument applied to E^c there is an open $V \supset E^c$ with $\mu(V \setminus E) < \epsilon/2$. Then $F = V^c$ is a closed subset of E and

$$\mu(U \setminus F) \leq \mu(U \setminus E) + \mu(V^c \cap E) < \epsilon.$$

This proves (a) and (b) is immediate: take a countable union with $\epsilon = 1/n$. $\square$

Thm 7.8: Let X be an LHC space in which every open set is σ-compact (this occurs if X is second countable). hen every Borel measure on X that is finite on compact sets is regular and hence Radon.

Proof. If μ is a Borel measure that is finite on compact sets, then $C_c(X) \subset L^1(\mu)$. Thus

$$I(f) = \int f d\mu$$

is a positive linear functional on $C_c(X)$ so is given by an associated Radon measure ν according to Theorem 7.2. We want to show $\mu = \mu$.

If $U \subset X$ is open, let $U = \cup K_j$ where K_j are compact. Choose f_1 continuous and compactly supported so that $f_1 \prec U$ $f_1 = 1$ on K_1.

Proceeding inductively, for $n > 1$ choose $f_n \in C_c(X)$ so that $f_n \prec U$ and $f_n = 1$ on $\cup_1^n K_j$ and also on $\cup_1^{n-1}\mathrm{supp}(f_j)$. Then f_n increases pointwise to χ_U so by the monotone convergence theorem

$$\mu(U) = \lim \int f_n d\mu = \lim f_n d\nu = \nu(U).$$

Thus $\mu = \nu$ on open sets.

For any Borel set E there is an open $V \supset E$ and a closed $F \subset E$ with $\nu(V \setminus F) < \epsilon$. But $V \setminus F$ is open, so $\mu(V \setminus F) = \nu(V\ F) < \epsilon$. In particular, $\mu(V) \leq \mu(E) + \epsilon$, so μ is outer regular.

Also $\mu(F) \geq \mu(E)\epsilon$, and F is σ-compact (since X is) so we deduce μ is inner regular. Thus μ is regular, and must equal ν by the uniqueness in the Riesz Representation Theorem. $\square$

Prop. 7.9: If μ is a Radon measure on X, then $C_c(X)$ is dense in $L^p(u)$ for $1 \leq p < \infty$.

Proof. Since the L^p simple functions are dense in L^p (Proposition 6.7), it is enough to show to show that for any Borel set E of finite measure, χ_E can be approximated in the L^p norm by elements of $C_c(X)$.

Given $\epsilon > 0$, by Proposition 7.5, we can choose a compact $K \subset E$ and an open $U \supset E$ so that $\mu(U \setminus E) < \epsilon$. By Urysohn's lemma we can choose $f \in C_c(X)$ such that $\chi_K \leq f \leq \chi_U$ Then

$$\|\chi_E - f\|_p \leq \mu(U \setminus F)^{1/p}, \epsilon^{1/p}.$$

. $\square$

Lusin's Theorem: Suppose that μ is a Radon measure on X and $f : X \to \mathbb{C}$ is a measurable function that vanishes outside a set of finite measure. Then for any $\epsilon > 0$ there exists $\phi \in C_c(X)$ such that $\phi = f$ except on a set of measure $< \epsilon$. If f is bounded, ϕ can be taken to satisfy $\|\phi\|_u \leq \|f\|_u$.

Nikolai Nikolaevich Luzin (1883–1950)

Dimitri Fedorovich Egorov (1869–1931)

Mikhail Yakovlevich Suslin (1894–1919)

Proof of Lusin's theorem. Let $E = \{x : f(x) \neq 0\}$, and suppose, to begin with, that f is bounded. Then $f \in L^1$ so by the previous result there is a sequence in $C_c(X)$ converging to f in the L^1 norm and hence a subsequence converging to f almost everywhere (Prop 2.32). By Egorov's theorem there is a set $A \subset E$ with $\mu(E \subset A) < \epsilon/3$ and $g_n \to f$ uniformly.

Choose $B \subset A$ compact and $U \supset E$ open so $\mu(A \setminus B) < \epsilon 3$ and $\mu(U \setminus E) < \epsilon 3$. Since $g_n \to f$ uniformly on B, f is continuous on B, so by Prop 4.34 there is $h \in C_c(X)$ with $h = f$ on B and $\text{supp}(h) \subset U$. But then $\{x : f(x) \neq h(x)\}$ is contained in $U \setminus B$, which has measure $< \epsilon$.

To complete the proof for f bounded, define $\beta : \mathbb{C} \to \mathbb{C}$ by $\beta(z) = z$ if $|z| \leq \|f\|_u$ and $\beta(z) = \|f\|_u \cdot \text{sgn(z)}$ otherwise (β is nearest point retraction onto the ball of radius $\|f\|_u$).

If $|z| > \|f\|_u$ and set $\phi = \beta \circ h$. Then $\phi \in C_c(X)$. Moreover, $\|\phi\|_u \leq \|f\|_u$ and $\phi = f$ on the set where $h = f$, so we are done if f is bounded.

If f is unbounded, let $A_n = \{x : 0 < |f(x)| \leq n\}$. Then A_n increases to E, so so $\mu((E \setminus A_n) < \epsilon/2$ for sufficiently large n. By the preceding argument there exists $\phi \in C_c(X)$ such that $\phi = f\chi_A$ except on a set of measure $< \epsilon/2$, and hence $\phi = fi$ except on a set of measure ϵ. $\square$

Defn: If X is a topological space, a function $f : X \to (-\infty, \infty]$ is called **lower semicontinuous (LSC)** if $\{x : f(x) > a\}$ is open for all $a \in \mathbb{R}$.

Defn: $f : X \to [-\infty, \infty)$ is called **upper semicontinuous (USC)** if $\{x : f(x) < a\}$ is open for all $a \in \mathbb{R}$.

]bf Prop. 7.11: Let$X$ be a topological space.

a. If U is open in X, then χ_U is LSC.

b. If f is LSC and $c \in [0, \infty)$, then cf is LSC.

b. If $\mathcal{G}$ is a family of LSC functions and $f(x) = \sup\{g(x) : g \in \mathcal{G}\}$, then f is LSC.

c. If f, g are LSC so is $f + g$.

d. If X is an LCH space and f is LSC and nonnegative, then

$$f(x) = \sup\{g(x) : g \in C_c(X), 0 \leq g \leq f\}.$$

Proof is left to the reader.

Prop 7.12: Let $\mathcal{G}$ be a family nonnegative LSC functions on LCH space X that is directed by (for every $g_1, g_2 \in \mathcal{G}$ there exists $g \in \mathcal{G}$ so that $g_1 \leq g$ and $g_2 \leq g$). Let $f = \sup\{g : g \in \mathcal{G}\}$. If μ is any Radon measure on X then

$$\int f d\mu = \sup\{\int g d\mu : g \in \mathcal{G}\}.$$

Cor. 7.13: If μ is Radon and $f \geq 0$ is LSC, then

$$\int f d\mu = \sup\{\int g d\mu : g \in C_c(X), 0 \leq g \leq f\}.$$

Prop 7.14: If μ is a Radon measure and f is a nonnegative Borel measurable function, then

$$\int f d\mu = \inf\{\int g d\mu : g \in C_c(X), g \geq f, g \text{ is LSC }\}$$

$$\int f d\mu = \sup\{\int g d\mu : g \in C_c(X), 0 \leq g \leq f, g \text{ is USC }\}$$

Chapter 7.3: The Dual of $C_0(X)$

Defn: $C_0(X)$ are the continues function on X that tend to zero at infinity, i.e., for any $\epsilon > 0$ there is a compact set K so that $|f(x)| < \epsilon$ for $x \in X \setminus K$.

$C_0(X)$ is the closure of $C_c(X)$ in the uniform norm.

If μ is a Radon measure on X the linear function

$$I(f) = \int_X f d\mu$$

extends from $C_c(X)$ to $C_0(X)$ iff it is bounded with respect to the uniform norm. This happens exactly when $\mu(X) < \infty$ because

$$\mu(X) = \sup\{\int f d\mu; f \in C_c(X), 0 \leq f \leq 1\}$$

by (7.3) and $|\int f d\mu| \leq \int |f| d\mu$.

Thus positive linear functionals on $C_0(X)$ are exactly integration against finite Radon measures.

What are the bounded linear functionals on $C_0(X)$? These will be linear combinations of positive functionals.

Lemma 7.15: If $I \in C_0(X, \mathbb{R})^*$, then there exist positive linear functionals $I^{\pm} \in C_0(X, \mathbb{R})^*$ so that $I = I^+ - I^-$.

Proof. For $f \in C_0(X, \mathbb{R})$ define

$$I^+(f) = \sup\{I(g) : g \in C_0(X, \mathbb{R}), 0 \leq g \leq f\}.$$

Obviously I^+ is a positive functional. Since $|I(g)| \leq \|g\|_u \cdot \|I\| \leq \|f\|_u \cdot \|I\|$ and $I(0) = 0$, we have

$$0 \leq I^+(f) \leq \|I\| \cdot \|f\|_u.$$

We claim I^+ is the restriction of a linear functional to $C_0(X,[0,\infty))$, i.e.,

$$I^+(cf) = cI^+(f), \qquad I^+(f_1) + I^+(f_2) = I^+(f_1 + f_2).$$

First note $I^+(cf) = cI^+(f)$ when $c \geq 0$. Also, $0 \leq g_k \leq f_k$, $k = 1, 2$ implies $g_1 + g_2 \leq f_1 + f_2$ so

$$I^+(f_1) + I^+(f_2) \geq I^+(f_1 + f_2).$$

If $0 \leq g \leq f_1 + f_2$, let $g_1 = \min(f_1, g)$ and $g_2 = g - g_1$. Then

$$0 \leq g_1 \leq f_1, \qquad 0 \leq g_2 \leq f_2$$

$$I(g) = I(g_1) + I(g_2) \leq I^+(f_1) + I^+(f_2)$$

$$I(f_1) + I(f_2) \leq I^+(f_1) + I^+(f_2)$$

so equality holds.

If $f \in C_0(X, \mathbb{R})$ then its positive and negative parts

$$f^+ = \max(f, 0), \qquad f^- = -\min(0, f),$$

are non-negative. Define $I^+(f) = I^(f^+) - I^-(f^-)$.

If $f = g - h$ where $g, h \geq 0$ then $g + f^- = h + f^+$, so

$$I^+(g) + I^+(f^+) = I^+(h) + I^+(f^+).$$

Thus

$$I^+(f) = I^+(g) - I^+(h),$$

and if follows that I^+ is linear on $C_0(X, \mathbb{R})$, i.e., if $h = f + g$ then

$I^+(h) = I^+(h^+) - I^+(h^-) = I^+(f^+) - I^+(f^-) + I^+(g^+) - I^+(g^-) = I^+(f)I^+(g).$

Next, note that

$$|I^+(f)| \leq \max(I^+(f^+), I^+(f')) \leq \|I\| \cdot \max(\|f^+\|_u, \|f'\|_u) \leq \|I\| \cdot \|f\|_u.$$

so that $\|I^+\| \leq \|I\|$.

Finally set $I^- = I - I^+$. We noted earlier that I^+ is a positive functional. If $f \geq 0$

$$\begin{aligned} I^-(f) &= I(f) - \sup\{I(g) : g \in C_0(X, \mathbb{R}), 0 \leq g \leq f\} \\ &= \ inf\{I(f-g) : g \in C_0(X, \mathbb{R}), 0 \leq g \leq f\} \\ &\geq \ 0 \end{aligned}$$

so I^- is also a positive functional.

$\square$

For any $I \in C_0(X, \mathbb{C})^*$ we can deduce

$$I(f) = \int f d\mu,$$

where

$$\mu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4)$$

are finite Radon measures.

Defn: A **signed Radon measure** is a real Borel measure whose positive and negative parts are Radon measures.

Defn: A **complex Radon measure** is a is a complex Borel measure whose real and imaginary parts are signed Radon measures.

On a second countable LCH space every complex Borel measure is Radon.

We define a norm on complex Radon measures by

$$\|\mu\| = |\mu|(X).$$

Prop. 7.16: If μ is a complex Borel measure, then μ is Radon iff $|mu|$ is Radon. Moreover, $M(X)$ is a vector space and $\mu \to \|\mu\|$ is a norm on it.

Proof. We observe that a finite positive Borel measure ν is Radon iff for every Borel set E and every $\epsilon > 0$ there exist a compact K and an open U such that $K \subset E \subset Ui$ and $\nu(U \setminus K) < \epsilon$, by Propositions 7.5 and 7.7.

If $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$ and $|\mu|(U \setminus K) < \epsilon$, then $\mu_j(U \setminus K) < \epsilon$, for $k = 1, 2, 3, 4$. So $|\mu|$ Radon implies all the μ_k are too.

If each μ_k is Radon, we can choose U_j, K_j so $\mu_j(U_j \setminus K_j) \leq \epsilon/4$, and hence $\mu(U \setminus K) \leq \epsilon$, where $U = \cup U_j$ and $K = \cap K_j$. Thus μ is Radon.

Same argument shows linearity. The norm property follows from Proposition 3.14, $|\mu_1 + \mu_2| \leq |\mu_1| + |\mu_2|$. $\square$

The Riesz Representation Theorem: Let X be an LCH space, and for $\mu \in M(X)$ and $f \in C_0(X)$ let $I_\mu(f) = \int f d\mu$. Then the map $\mu \to I_\mu$ is an isometric isomorphism from $M(X)$ to $C_0(X)^*$.

Proof. We have already shown that every linear functional on $C_0(X)$ is of the form I_μ, so we only have to check we have an isometry.

If $\mu \in M(X)$, then by Proposition 3.13c we have

$$|\int f d\mu| \leq \int |f| d|\mu| \leq \|f\|_u \cdot \|\mu\|.$$

so integration gives a linear functional with norm bounded by $\|\mu\|$.

On the other hand, if $h = d\mu/d|mu|$ then $|h| = 1$ by Proposition 3.13b, so by Lusin's theorem for any $\epsilon > 0$ there is a $f \in C_c(X)$ such that $\|f\|_u = 1$ and $f = \overline{h}$ except on a set E with $|\mu|(E) < \epsilon/2$. Then

$$\begin{aligned}
\|\mu\| &= \int |h|^2 d|\mu| \\
&= \int \overline{h} d|\mu| \\
&\leq |\int f d\mu| + |\int (f - \overline{(}h) d\mu| \\
&\leq |\int f d\mu| + |\int (f - \overline{(}h) d\mu| \\
&\leq |\int f d\mu| + 2|\mu|(E) \\
&< |\int f d\mu| + \epsilon \\
&\leq \|I_m u\| + \epsilon \quad \square
\end{aligned}$$

Cor 7.18: If X is a compact Hausdorff space, then $C(X)^*$ is isometrically isomorphic to $M(X)$.

If μ is a positive Radon measure on X and $f \in L^1(\mu)$ then the complex measure $f d\mu$ is easily seen to be Radon (Exercise 8) with norm $\|f\|_1$.

Thus $L^1(\mu)$ isometrically embeds in $M(X)$. The range is the set of measures absolutely continuous with respect to μ.

Defn: the **vague topology** on $M(X)$ is the same as the weak* topology (comes from probability theory).

Prop. 7.19: Suppose $\mu, \{\mu_n\} \subset M(X)$ and let $F_n(x) = \mu_n((-\infty x])$ and $F(x) = \mu((-\infty x])$.

(a) if $\sup \|\mu_n\| < \infty$ and $f_n \to F$ at every x where F is continuous, then $\mu_n \to \mu$ vaguely.

(b) If $\mu_n \geq 0$, $\mu_n \to \mu$ vaguely and

$$\limsup_{x \to -\infty} [\sup_n F_n(x)] = 0,$$

then $F_n(x) \to F(x)$ at every x where F is continuous.

See the text for the proof.