MAT 533, FALL 2021, Stony Brook University

REAL ANALYSIS II

FOLLAND'S REAL ANALYSIS: CHAPTER 5

Christopher Bishop

1. Wed, Feb 9, 2021

Chapter 5: Elements of Functional Analysis

Chapter 5.1: Normed Spaces

Vector spaces over $K = \mathbb{R}$ or $\mathbb{C}$.

If $x \in \mathcal{X}$, Kx is one dimensional space spanned by x.

$\mathcal{Y} + \mathcal{Z} = \{y + z : y \in \mathcal{Y}, z \in \mathcal{Z}\}$.

Defn: A **semi-norm** is a function $x \to \|x\| \in [9.\infty)$ so that

- $\|x + y\| \leq \|x\| = \|y\|$ (triangle inequality)
- $\|\lambda x\| = |\lambda|\|x\|$

Defn: A **norm** is a semi-norm so that $\|x\| = 0 \Rightarrow x = 0$.

Defn: A **normed vector space** is a vector space with a norm. It is a metric space.

Defn: A complete normed vector space is a **Banach space**.

Stefan Banach (1892–1945)

Banach Center

Institute of Mathematics

Polish Academy of Sciences

Examples:

- $\mathbb{R}^n$ with usual norm.

- $L^p(d\mu) = \mu$-measurable functions with semi-norm

$$\|f\|_p = \left(\int |f|^p d\mu \right)^{1/p}.$$

Make it a norm by taking equivalence classes of functions $f \sim g$ if $f = g$ almost everywhere.

- $L^\infty =$ essentially bounded measurable functions with semi-norm

$$\|f\|_\infty = \inf\{\alpha : \mu(\{x : |f(x)| > \alpha\}) = 0\}$$

This is also a Banach algebra (closed under multiplication).

- $C(X)$: continuous functions on a compact space X

$$\|f\| = \sup_{x \in X} |f(x)|.$$

This is also a Banach algebra.

• $C_c(X)$: compactly supported continuous functions on a non-compact space X.

• $BC(X)$: bounded continuous functions on a non-compact space X.

• $C_0(X)$: continuous functions that tend to 0 at infinity on a non-compact space X.

• Special cases:

ℓ_p consists of sequences so that $\sum_m |a_n|^p < \infty$.

ℓ_∞ consists of bounded sequences.

c_0 sequences that tend to zero.

• Finite Borel measures on a compact space with $\|\mu\| = |\mu(X)|$ (the absolute value of the measure).

- Lipschitz functions: continuous functions so that

$$\|f\| = \|f(x_0)\| + \sup_{x,y} \frac{|f(x) - f(y)|}{|x - y|} < \infty.$$

- Hölder functions: give $0 < \alpha \leq 1$ continuous functions so that

$$\|f\| = \|f(x_0)\| + \sup_{x,y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty.$$

- $C^{1,\alpha}(\mathbb{R}^n)$: continuously differentiable functions whose derivative is α-Hölder.

- Sobolev spaces: W^1, p, $f \in L^p$ and $f' \in L^p$. Define on smooth functions then take completion.

- Sobolev space $H^{1/2}$ (half a derivative in L^2):

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x) - f(y)|^2}{|x - y|^2} dxdy < \infty.$$

- Bounded Mean Oscillation (BMO): for f measurable on $\mathbb{R}$ let

$$m_I(f) = \frac{1}{|I|}\int_I f(x)dx,$$

denote mean value of f on I.

$$\|f\|_{\text{BMO}} = \sup_I m_I(f - m_I(f))).$$

In other words, on every interval, f is close to its mean value on that interval.

For bounded intervals, $L^p \subset \text{BMO} \subset L^\infty$ for all $p < \infty$. BMO is a good substitute for L^∞ in many situations (e.g., studying harmonic conjugation bounded on L^p and BMO but not on L^∞.

- Vanishing Mean Oscillation (VMO): subspace of BMO so that

$$m_I(f - m_I(f))) \to 0 \text{ as } |I| \to 0.$$

Is closure of continuous functions in BMO.

- $H^\infty(\mathbb{D})$: bounded holomorphic functions on $\mathbb{D}$ with sup norm. Is complete because uniform limit of holomorphic functions is holomorphic.

- Hardy Spaces $H^p(\mathbb{T})$: holomorphic on disk and
$$\sup_{0<r<1} \int_{\mathbb{T}} |f(rie^{i\theta})|^p d\theta < \infty.$$
Equals subspace of $L^p(\mathbb{T}, d\theta)$ that has holomorphic extension to disk (in a certain sense).

- Bergman spaces: holomorphic functions on disk in $L^p(dxdy)$:
$$\int_{\mathbb{D}} |f(z)|^p dxdy < \infty.$$

- Dirichlet space: holomorphic on disk and
$$\int_{\mathbb{D}} |f'(z)|^2 dxdy < \infty.$$
Holomorphic functions whose images have finite area, counted with multiplicity.

• Orlicz space: measurable functions so that

$$\int |f(x)| \log^+ |f(x)| d\mu(x) < \infty.$$

or more generally

$$\int \Phi(|f(x)|) d\mu(x) < \infty$$

where Φ is convex and $\Phi(x)/x \to \infty$ as $x \to \infty$ and $\to 0$ as $x \to 0$. This is not a norm. To get a norm let $\Psi =_0^x \Phi^{-1}(t)dt$ and define

$$\|f\|_\Phi = \sup\{\int fg d\mu : \int \Psi(g) d\mu \le 1\}.$$

• Lorentz space: measurable functions with semi-norm

$$\|f\|_{p,q} = p^{1/q} \left(\int_0^\infty t^q \mu(\{x : |f(x)| \ge t\})^{q/p} \frac{dt}{t} \right)^{1/q}.$$

Defn: Two norms are **equivalent** if there is $C_1, C_2 < \infty$ so that

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_2.$$

Defn: a series $\sum_{k=1}^{\infty} x_n$ in a normed vector space **converges** if $s_n = \sum_{k=1}^{n} x_n$ has a limit.

Defn: a series is **absolutely convergent** if $\sum \|x_n\| < \infty$ (a sum of real numbers).

Theorem 5.1: A normed vector space $\mathcal{X}$ is complete iff every absolutely convergent series in $\mathcal{X}$ converges.

Proof. Suppose $\mathcal{X}$ is complete and $\sum x_n$ is absolutely convergent. Let $s_n = \sum_{k=1}^{n} x_n$. Then

$$\|s_n - s_m\| \leq \sum_{k=n}^{m} \|x_k\| \to 0,$$

so $\{s_n\}$ is Cauchy and has a limit.

Conversely, assume absolutely convergent series converge. Let $\{x_n\}$ be Cauchy. WLOG we may pass to a subsequence (also called x_n) so that $\|x_n - x_m\| \le 2^{-n}$ for $m > n$. If

$$y_1 = x_1, \qquad y_n = x_{n+1} - x_n$$

Then

$$\sum_1^n y_k = x_n,$$

and $\sum y_k$ is absolutely convergent, so has a limit z, which is also limit of x_n. $\square$

Product spaces:

If $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces, $\mathcal{X} \times \mathcal{Y}$ becomes a normed vector space with

$$\|(x, y)\| = \max(\|x\|, \|y\|).$$

Can also use

$$\|(x, y)\| = \|x\| + \|y\|$$
$$\|(x, y)\| = \left(\|x\|^2 + \|y\|^2\right)^{1/2}.$$

or other equivalent norms.

Quotient spaces:

If $\mathcal{Y}$ is vector subspace of $\mathcal{X}$ define $x \sim y$ if $x - y \in \mathcal{Y}$.

Set of equivalence classes is a vector space denoted $\mathcal{X}/\mathcal{Y}$, the quotient space.

If $\mathbf{y}$ is closed we can define quotient norm

$$\|x + \mathcal{Y}\| = \inf_{y \in \mathcal{Y}} \|x + y\|$$

.

Defn: A linear map $T : \mathcal{X} \to \mathcal{Y}$ between normed vector spaces is **bounded** if there there exists $C < \infty$ so that

$$\|Tx\| \leq C\|x\|$$

Prop. 5,2: If $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces and $T : \mathcal{X} \to \mathcal{Y}$ is linear then TFAE:

(a) T is continuous.

(b) T is continuous at 0.

(c) T is bounded.

Proof. (a) $\Rightarrow$ (b) is trivial.

To prove (b) $\Rightarrow$ (c), note there is a neighborhood U of 0 in $\mathcal{X}$ so that $T(U)$ is inside the unit ball of $\mathcal{Y}$. Since U is open it contains a δ-ball around the origin. So $\|x\| < \delta \Rightarrow \|Tx\| < 1$. Thus in general,

$$\|Tx\| = \frac{1}{\delta}\|x\| \cdot \|T(\delta x/\|x\|)\| \leq \frac{1}{\delta}\|x\|.$$

To prove (b) $\Rightarrow$ (c), note that

$$\|Tx - Ty\| \leq C\|x - y\|,$$

so T is continuous, even Lipschitz. $\square$

Defn: $L(\mathcal{X},\mathcal{Y})$ is vector space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. It is a normed space wit

$$\begin{aligned}\|T\| &= \sup\{\|Tx\| : \|x\| \leq 1\}\\ &= \sup\{\|Tx\|/\|x\| : x \in \mathcal{X}\setminus\{0\}\}\\ &= \{\|Tx\|/\|x\| : x \in \mathcal{X}\setminus\{0\}\}.\end{aligned}$$

Prop. 5.4 If $\mathcal{Y}$ is complete, so is $L(\mathcal{X},\mathcal{Y})$.

Proof. Suppose $\{T_n\}$ is Cauchy. Then for a fixed x, T_nx is also Cauchy since

$$\|T_nx - T_mx\| \leq C\|T_n = T_m\|\|x\|,$$

so $T_nx \to Tx \in \mathcal{Y}$. Note that

$$T(\lambda x) = \lim_n T_n(\lambda x) = \lim_n \lambda T_n(x) = \lambda T(x),$$

and

$$T(x+y) = \lim_n T_n(x+y) = \lim_n T_n(x+y) = \lim_n T_n(x)+T_n(+y) = \lambda T(x)+T(x),$$

so T is linear.

If $\|x\| \leq 1$, then

$$\|T(x)\| = \|\lim_n T_n(x)\| \leq \lim_n \|T_n\|,$$

so T is bounded. Moreover, if $\|x\| \leq 1$,

$$\|(T - T_n)(x)\| = \lim_m \|T_m x - T_n x\| \leq \sup_{m>n} \|T_m - T_n\|,$$

is as small as we wish if n is large, since $\{T_n\}$ is Cauchy. $\square$

Operator norms satisfy

$$\|TS\| \leq \|T\| \cdot \|S\|.$$

If $\mathcal{X}$ is a Banach space this makes $L(\mathcal{X}, \mathcal{X})$ into a Banach algebra.

Defn: A bounded operator $T \in L(\mathcal{X}, \mathcal{Y})$ is **invertible** if it is 1-1, onto and T^{-1} is bounded. Also called an **isomorphism**.

If we set

$$Tf(y) =\rightarrow \int_0^x f(y)dy$$

then T is bounded 1-1 operator from $L^1([0,1])$ to $C([0,1])$, but is not onto.

We shall see later that a bijective bounded operator between Banach spaces is automatically invertible (open mapping theorem).

5.2 Linear Functionals

Defn: a linear map from $\mathcal{X}$ to the scalars is called a **linear functional**.

Defn: the collection of bounded linear maps $L(\mathcal{X}, K)$ is called the **dual space** of $\mathcal{X}$.

Examples:

$(L^1)^* = L^\infty$ but $(L^\infty)^* \neq L^1$ in general

$$L_f(g) = \int fgd\mu,$$

$(L^p)^* = L^q$ where $\frac{1}{p} + \frac{1}{q} = 1$, 1 ¡p,q, ¡ infty

$(L^2)^* = L^2$

$C(X)^* =$ finite measures on X

Prop. 5.5: Let $\mathcal{X}$ be a vector space over $\mathbb{C}$. If f is a complex linear functional on $\mathcal{X}$ and and $u = \mathrm{Re} f$, then u is a real linear functional, and $f(x) = u(x) - iu(ix)$ for all $x \in \mathcal{X}$.

Conversely, if u u is a real linear functional on $\mathcal{X}$ and $f(x) = u(x) - iu(ix)$, then f is complex linear. If X is normed, $\|u\| = \|f\|$.

Proof. If f is complex linear, $u = \mathrm{Re} f$, and λ is real, then

$u(\lambda x+y) = \mathrm{Re} f(\lambda x+y) = \mathrm{Re}\lambda f(x)+f(y) = \lambda \mathrm{Re} f(x)+\mathrm{Re} f(y) = \lambda u(x)+u(y)$,

so u is real linear.

Also,

$$\mathrm{Im}(f(x) = -\mathrm{Re}(if(x)) = -u(ix),$$

so

$$f(x) = \mathrm{Re} f(x) + i\mathrm{Im} f(x) = u(x) - iu(ix).$$

Suppose u is real linear, set $f(x) = u(x) - iu(ix)$. Then f is clearly linear over $\mathbb{R}$ since u is, and

$$f(ix) = u(ix) - iu(-x) = u(ix) + iu(x) = i(u(x) - iu(ix)) = if(x).$$

This implies f is linear over $\mathbb{C}$.

If $\mathcal{X}$ is normed then

$$|u(x)| = |\mathrm{Re} f(x)| \leq |f(x)|,$$

implies $\|u\| \leq \|f\|$.

Conversely, if $f(x) \neq 0$, then let $\alpha = \overline{f(x)/|f(x)|}$. Then $f(\alpha x) = \alpha(x)$ is real so $f(\alpha x) = u(\alpha x)$, so

$$|f(x)| = \alpha f(x) = f(\alpha x) = u(\alpha x) \leq \|u\||\alpha|\|x\| = \|u\|\|x\|.,$$

Thus $\|f\| \leq \|u\|$. $\square$

Defn: a **sublinear functional** on $\mathcal{X}$ is a map $p : \mathcal{X} \to \mathbb{R}$ so that

$$p(x+y) \leq p(x) + p(y), x, y \in \mathcal{X}$$

$$p(\lambda x) = \lambda p(x), x \in \mathcal{X}, \lambda > 0.$$

It is not obvious that linear functionals exist, but they always do.

The Hahn-Banach Theorem: Suppose $\mathcal{X}$ is a real vector space, p a sublinear functions on $\mathcal{X}$, $\mathcal{M}$ a subspace of $\mathcal{X}$ and f a linear functional on $\mathcal{M}$ such that $f(x) \leq p(x)$ for all $x \in \mathcal{M}$. Then there is a linear functional F on $\mathcal{X}$ that extends f and $F(x) \leq p(x)$ for all $x \in \mathcal{X}$.

Proof. First we extend by one dimension, from f on $\mathcal{M}$ to g on $\mathcal{M}+\mathbb{R}x$, $x \notin \mathcal{M}$.

If $y_1, y_2 \in \mathcal{M}$, then $g(y) \leq p(y)$ implies

$$f(y_1)+f(y_2)=f(y_1+y_2) \leq p(y_1+y_2) \leq (y_1-x)+p(x+fy_2),$$

or

$$f(y_1)-p(y_1-x) \leq p(x+y_2)-f(y_2).$$

Note that all the y_1's are on one side and the y_2's on the other. Hence for any x,

$$\sup_{y_1} f(y_1)-p(y_1-x) \leq \inf_{y_2} p(x+y_2)-f(y_2).$$

Choose some α between these two real numbers (possibly equal).

Define

$$g(y + \lambda x) = f(y) + \alpha x.$$

This is linear on $\mathcal{M} + \mathbb{R}x$ and extends f . Just need to check $g \leq p$.

First assume $\lambda > 0$. Then since α is less than the RHS above,

$$\begin{aligned} g(y + \lambda x) &= \lambda(f(y/\lambda) + \alpha) \\ &\leq \lambda(f(y/\lambda) + p(x + (y/\lambda)) - f(y/\lambda)) \\ &\quad \lambda p(x + (y/\lambda)) \\ &\quad p(\lambda x + y) \end{aligned}$$

Next assume $\lambda = -\mu < 0$. Then since α is larger than the LHS above,

$$\begin{aligned} g(y + \lambda x) &= \mu(f(-y/\mu) - \alpha) \\ &\leq \mu(f(y/\mu) + p(-x + y/\mu) - f(y/\mu)) \\ &\quad \mu p(-x + y/\mu) \\ &\quad p(\lambda x + y) \end{aligned}$$

The family of linear extensions dominated p is partially ordered, and the union of nested subspaces is a subspace, so by Zorn's Lemma there is a maximal element. If this element is not defined on the whole space $\mathcal{X}$, then the argument about extends it to a larger subspace, contradicting maximality. Thus every element extends. $\square$

Hans Hahn (1879–1934)

If p is a semi-norm and f is a linear functional then $f \leq x$ iff $|f| \leq p$ because $p(x) = p(-x)$.

The Complex Hahn-Banach Theorem: Let $\mathcal{X}$ be a complex vector space, p a seminorm on $\mathcal{X}$, $\mathcal{M}$ a subspace of $\mathcal{X}$, and f a complex linear functional on $\mathcal{M}$ such that $|f(x)| \leq p(x)$ for $\in xM$. Then there exists a complex linear functional F on $\mathcal{X}$ such that $|F(x)| \leq p(x)$ for all $\in \mathcal{X}$ and extending f.

Proof. Let $u = \mathrm{Re} f$. Extend u to U by the real version on Hahn-Banach and then set $F(x) = U(x) - iU(ix)$ which is a complex linear extension of f. Finally, if $\alpha = \overline{F(x)/|F(x)|}$ then

$$|F(x)| = \alpha F(x) = F(\alpha x) = U(\alpha x) \leq p(\alpha x) = p(x).$$

$\square$

Theorem 5.8: Let $\mathcal{X}$ be a normed vector space.

(a) If $\mathcal{M}$ is a closed subspace of $\mathcal{X}$ and $\in \mathcal{X} \setminus \mathcal{M}$, then there exists $\in \mathcal{X}^*$ such $f(x) \neq 0$ and $f|_{\mathcal{M}} = O$. In fact, if

$$\delta = \text{dist}(x, \mathcal{M}) = \inf_{y \in \mathcal{M}} ||x - y\|,$$

then f can be taken to satisfy $\|f\| = 1$ and $f(x) = \delta$.

(b) If $x \neq 0$, there exists $f \in \mathbf{x}^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.

(c) The bounded linear functionals on $\mathcal{X}$ separate points.

(d) If $x \in \mathcal{X}$, define $\hat{x} : \mathcal{X}^* \to \mathbb{C}$ by $\hat{x}(f) = f(x)$. Then the map $x \to \hat{x}$ is a linear isometry from $\mathcal{X}$ into $\mathcal{X}^{**}$ (the double dual $(\mathcal{X}^*)^*$).

Proof. **Proof of (a):** Define f on $\mathcal{M} + \mathbb{C}x$ by

$$f(y + \lambda x) = \lambda\delta.$$

Then $f(x) = \delta$, f is zero on $\mathcal{M}$ and for $\lambda \neq 0$

$$|f(y + \lambda x)| = |\lambda|\delta \leq |\lambda| \cdot \|y/\lambda + x\| \leq \cdot \|y + \lambda x\|,$$

so $|f(z)| \leq |z|$ on the extension. Hahn-Banach can now be applied to $\mathcal{M} + \mathbb{C}x$.

Proof of (b): Special case of (a) when $\mathcal{M} = \{0\}$.

Proof of (c): if $x \neq y$, there exists $f \in \mathcal{X}^*$ with $f(x - y) \neq 0$.

Proof of (d): Easy to check $\hat{x}$ is a linear functional on $\mathcal{X}^*$ and the map $\to \hat{x}$ is linear. Moreover,

$$\|\hat{x}(f)| = |f(x)| \leq \|f\| \cdot \|x\|,$$

so $\|\hat{x}\| \leq \|x\|$. On the other hand, (b) implies that $\|\hat{x}\| \geq \|x\|$. $\square$

We may consider $\mathcal{X} \subset \mathcal{X}^{**}$.

Defn: If $\mathcal{X} = \mathcal{X}^{**}$ we say $\mathcal{X}$ is **reflexive**.

Non-reflexive example: $(c_0)^* = \ell_1$ and $(c_0)^{**} = (\ell_1)^* = \ell_\infty$.

Theorem (3.3, Lax's book): Suppose $\mathcal{X}$ is a real linear space and $\mathcal{A}$ is a collection of commuting linear maps $X \to X$, and p is a sub-additive function so that $p(Ax) = p(x)$ for all $A \in \mathcal{A}$. Suppose f is a linear functional defined on a linear subspace $\mathcal{Y}$ so that

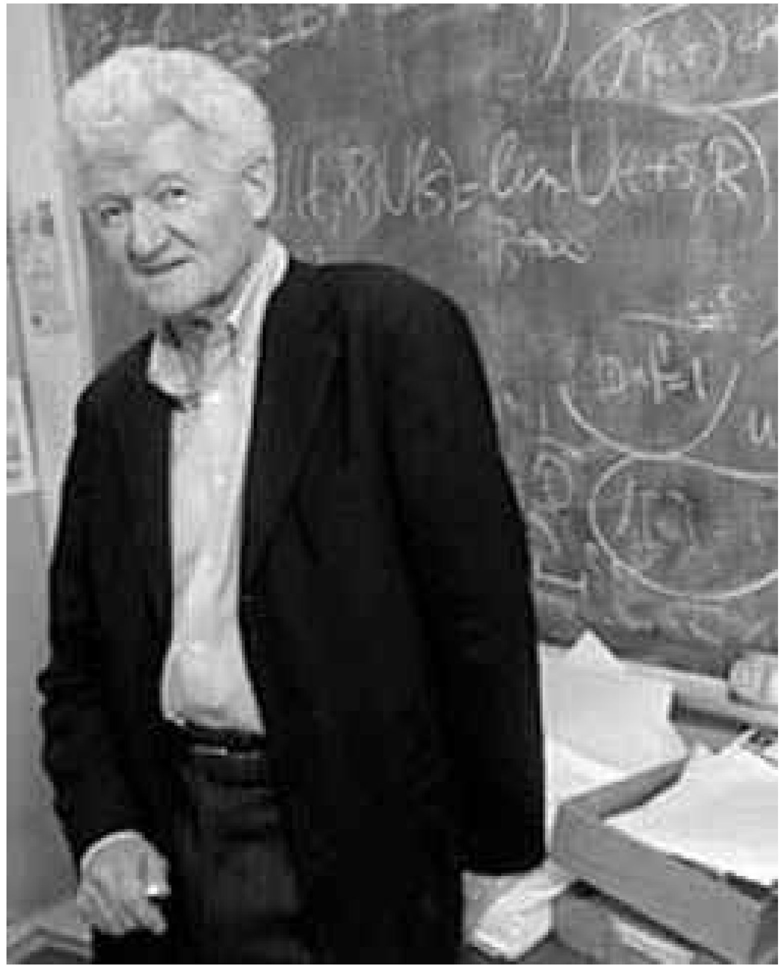
(a) f is dominated by p

(b) Y is invariant under each element of $\mathcal{A}$.

(c) f is invariant, i.e., $f(Ax) = f(x)$, for all $A \in \mathcal{A}$.

Then f has an extension to $\mathcal{X}$ that is dominated by o and also $\mathcal{A}$ invariant.

Proof from Lax's Book "Functional Analysis"



Peter Lax (1926–present)

Examples:

Banach Limits: linear functional on ℓ_∞ that are shift invariant and extend taking limits on the linear subspace of sequences that have limits.

Lebesgue integration defines a translation invariant linear functional on the space of bounded measurable functions on the unit circle. Hahn-Banach extends it to a rotation invariant finitely additive measure on all subsets of the circle. See Lax's book, Chapter 4.

There are no countable additive, rotation invariant measures. There is no finitely additive, rotational invariant measure on the 2-sphere: rotations of 3-space do not commute, so the lower dimensional proof does not extend. Non-existence is based on the Banach-Tarski paradox.

The Hahn-Banach theorem surveyed by Gerard Buskes,1993.

Alfred Tarski (1901–1983)

5.3 The Baire Category Theorem and its Consequences

The Baire Category Theorem: Let X be a complete metric space.

(a) If $\{U_n\}_1^\infty$ are open dense subsets of X, then $\cap_n U_n$ is dense in X.

(b) X is not a countable union of nowhere dense sets.

Proof. **Proof of (a):** Given W nonempty and open, we must show $W \bigcap \cap_n U_n \neq \emptyset$. Since $U_1 \cap W$ is open and nonempty, it contains a ball $B(r_0, x_0)$, and we can assume that $0 < r_0 < 1$. For $n > 0$, choose x_n and $r_n < 2^{-n}$ so that

$$\overline{B(x_n, r_n)} \subset U_n B(x_{n-1}, r_{n-1}).$$

Then $\{x_n\}$ is Cauchy and the limit point x is in

$$\overline{B(x_n, r_n)} \subset U_n$$

for all n, hence in $U_n \cap W$, as desired.

Proof of (b): If $\{E_n\}$ is a sequence of nowhere dense sets in X, then their complements is a sequence of open dense sets. By (a) the intersection of the complements is non-empty so the union of the $E_n's$ is not everything. $\square$

It suffices for X to be homeomorphic to a complete metric space.

Defn: A **Polish space** is a separable topological space, that has a compatible metric making it complete.

Defn: $E \subset X$ is **first category** if E is a countable union of nowhere dense sets. Otherwise E is **second category**

Defn: **meager** is same as first category. A set is **residual** if it is the complement of a meager (first category) set.

Residual and second category are not the same.

René-Louis Baire (1874–1932)

Category arguments are often used to give existence proof, by showing that "most" elements of a space have a certain property.

Example: nowhere differentiable functions on $[0, 1]$ are residual in $C([0, 1])$.

On the real line, we often use measure instead, e.g., "almost every point has property P", but there are usually no invariant measure on a infinite dimensional space like $C([0, 1])$.

Example: A meager subset of $\mathbb{R}$ can have full measure.

There is a measure theoretic notion of "measure zero". If $\mathcal{X}$ is a separable, infinite dimensional Banach space then we say a Borel subset A is **prevalent** if there is Borel probability measure μ on $\mathcal{X}$ so that $\mu(A + x) = 1$ for every $x \in \mathcal{X}$. A set is **negligible** if its complement is prevalent.

Example: the set of nowhere differentiable functions on $[0, 1]$ is prevalent in $C([0, 1])$. The measure μ is Wiener measure; this fact corresponds to the theorem that for any continuous function f, $B + f$ is almost surely nowhere differentiable , where B is Brownian motion.

See Section 6.5 (page 176) of my book Fractals in probability and analysis

Defn: A map $f : X \to Y$ is called **open** if it maps open sets to open set. If f is invertible, its inverse is then continuous.

The Open Mapping Theorem: Let **x** and $\mathcal{Y}$ be Banach spaces. If $T \in L(\mathcal{X}, \mathcal{Y})$ is surjective, then T is open.

Proof. We may assume $\|T\| = 1$. Let $B_X(r)$ denote the (open) ball of radius r around $0 \in \mathcal{X}$. If suffice to show the image of $B - X(r)$ contains a ball around 0 in Y. $X = \bigcup_n B_X(n)$ and T is surjective, we have $\mathcal{Y} = \bigcup T(B_X(n)$. By Baire's theorem some $T(B_X(n))$ is somewhere dense, hence so is $T(B_X(1))$, by linearity.

Choose $r > 0$ so small that $\overline{B_Y(y_0, 4r)}$ is a closed ball in $T(B_X(1))$.

Choose $y_1 = Tx_1 \in T(B_X(1))$ so that $\|y_1 - y_0\| \leq 2r$. Then

$$T(B_X(y_1, 2r)) \subset B_Y(T(y_1), 2r) \subset B_Y(y_0, 4r) \subset T(B_X(1))$$

and hence

$$y = -Tx_1 + (y + y_1) \in \overline{T(-x_1 + B_X(1))} \subset \overline{T(B_X(2))}.$$

Thus (dividing by 2) $\|y\| < r \Rightarrow T(y) \in \overline{T(B_X(1))}$.

In other words $T(B_X(1))$ is dense $B_Y(r)$. (or $T(B_X(1/r))$ is dense $B_Y(1)$).

Dilating, it follows that if $\|y\| < r2^{-n}$, then $y \in \overline{T(B_{2^{-n}})}$.

Suppose $\|y\| < r/2$. We can find $x_1 \in B_X(1/2)$ such that $\|y - T(x_1)\| < r/4$. By induction, we can find x_n with norm $< 2^{-n}$ so that

$$\|y - \sum_1^n x_k\| < r2^{-n}.$$

$\mathcal{X}$ is compete, so $\sum_1^\infty x_n \to x$ with $\|x\| \leq \sum 2^{-n} \leq 1$ and $y = Tx$. $\square$

Corollary of Proof: If $T : X \to Y$ is bounded and there are $\epsilon > 0$ and $R < \infty$ so that $T(B_X(R))$ is $(1-\epsilon)$-dense in $B_Y(1)$, then $T(X) = Y$ (T is onto).

Application to Interpolating sequences: Every $f \in L^\infty(\mathbb{T})$ as a harmonic extension u on disk. Given a sequence $\{z_n\} \subset \mathbb{D}$ we can restrict u to $\{z_n\}$ to be bounded linear map $L^\infty(\mathbb{T}) \to \ell^\infty$. When is it onto?

For example this happens if $z_n = 1 - 4^{-n}$. Enough to show there are functions f_n with disjoint supports on $\mathbb{T}$ so that $0 \le f \le 1$

$$u_n(z_n) \ge \epsilon, \text{ for all } n$$

$$\sum_{k \ne n} u_k(z_n) \ge 1 - \epsilon, \text{ for all } n.$$

This is not too hard to do. In general, a sequence in $\mathbb{D}$ is interpolating iff it is separated in hyperbolic metric and $\sum 1 - |z_n| < \infty$.

Cor. 5.11: Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. If $T \in L(\mathcal{X}, \mathcal{Y})$ is bijective, then T^{-1} is continuous, i.e., T is an isomorphism.

Defn: If $T \in L(\mathcal{X}, \mathcal{Y})$, the **graph** of T is the set

$$\Gamma(T) = \{(x, y) \in \mathcal{X} \times \mathcal{Y} : y = Tx\}$$

The graph is a linear space. If $\mathcal{X}$ and $\mathcal{Y}$ are complete, so is $\mathcal{X} \times \mathcal{Y}$. Hence is Γ is complete iff it is closed.

The graph of a continuous function is closed, but not conversely (e.g., $f(x) = 1/x$ for $x \neq 0$, $f(0) = 0$).

But for linear maps, the converse is true.

The Closed Graph Theorem: If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $T : \mathcal{X} \to \mathcal{Y}$ is a closed linear map, then T is bounded.

Proof. Let π_1 and π_2 be the projections of $\Gamma(T)$ onto the two coordinates. These are bounded linear maps onto $\mathcal{X}$ and $\mathcal{Y}$ respectively. Since π_1 is $1-1$ and onto, it is invertible, hence continuous. Thus

$$T = \pi_2 \circ \pi_1^{-1}$$

is also continuous. $\square$

Continuity = if $x_n \to x$ then $Tx_n \to Tx$.

Closedness = if $x_n \to x$ and $Tx_n \to y$ then $Tx_n \to y$.

We get to assume more in the second case, so it is often easier to prove the conclusion. It is often easier to prove what limit has to be, if it exists, than proving the limit does exist.

For example if $f : X \to X$ is continuous and $f^n(x)$ has a limit y then

$$f(y) = f(\lim_n f^n(x)) = \lim_n f^{n+1}(x) = \lim_n f^n(x) = y,$$

so y must be a fixed point. Hard part is to show limit exists.

Nice application of Closed Graph theorem:

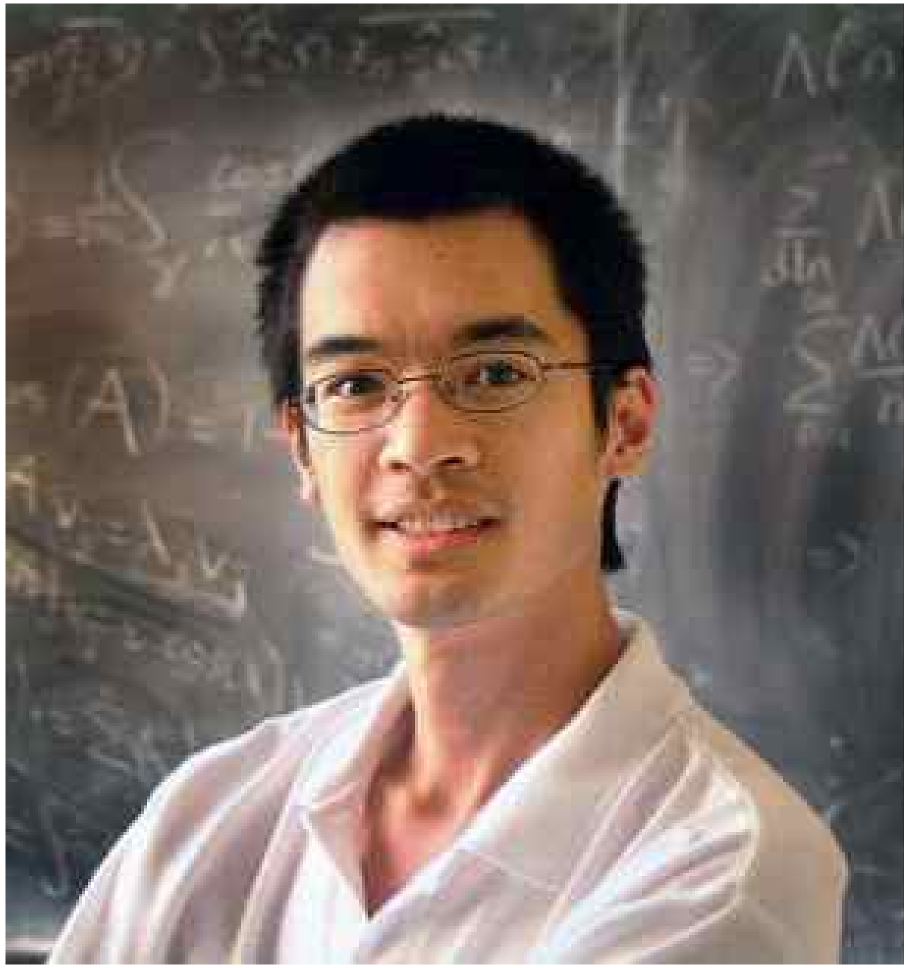
Thm: Let T be a bounded operator on the Hilbert space $L^2([0,1])$ so that if f is a continuous function so is Tf. Then the restriction of T to $C([0,1])$ is a bounded operator.

Proof by de Lamadrid, 1963

Proposition: $V = (V, \mathcal{F})$ be a Hausdorff topological vector space. Then, up to equivalence of norms, there is at most one norm $\|\cdot\|$ one can place on V so that $(V, \|\cdot\|)$ is a Banach space whose topology is at least as strong as $\mathcal{F}$. In particular, there is at most one topology stronger than $\mathcal{F}$ that comes from a Banach space norm.

Proof on Terry Tao's blog

Another Terry Tao blog on closed graph theorem analogs in different parts of mathematics



Terry Tao (1975–present)

Application: If $\mathcal{H}$ is a Hilbert space and $M : \mathcal{H} \to H$ satisfies

$$\langle Mx, y\rangle = \langle x, My\rangle,$$

is called **symmetric**.

Lemma (Hellinger and Toeplitz): A symmetric operator defined on whole space must be bounded.

Proof. Suppose $x_n \to x$ and $Mx_n \to u$. Then

$$\langle u, y\rangle = \lim_n \langle Mx_n, y\rangle = \lim_n \langle x_n, My\rangle = \langle x, My\rangle = \langle Mx, y\rangle.$$

Since the dual separates points $u = x$. Hence M is closed, hence bounded by Closed Graph Theorem.

Example: The Laplace transform, for $f \in L^2(0, \infty)$,

$$g(s) = Lf(s) = \int_0^\infty f(t)e^{-st}dt.$$

$$\int Lf(s)g(s)ds = \int f(s)Lg(s)ds = \int_0^\infty \int_0^\infty f(t)g(s)e^{-st}dsdt.$$

A more careful calculation shows $\|L\| = \sqrt{\pi}$.

$L^2 = L \circ L$ is the Hilbert-Hankel operator:

$$g(r) = \int_0^\infty \frac{f(t)}{t+r}dt.$$

This is bounded with norm π. It is bounded on L^p for $1 < p < \infty$, but L is not (except $p = 2$).

Ernst Hellinger (1883–1950)

Otto Toeplitz (1881-1940)

Hermann Hankel (1837-1873)

The Uniform Boundedness Principle: Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces and $\mathcal{A} \subset L(\mathcal{X}, \mathcal{Y})$.

(a) If $\sup_{T\in\mathcal{A}} \|Tx\| < \infty$ for all x in some non-meager subset of $\mathcal{X}$, then $\sup_{T\in\mathcal{A}} \|T\| < \infty$.

(b) If $\mathcal{X}$ is a Banach space and $\sup_{T\in\mathcal{A}} \|Tx\| < \infty$ for all x in $\mathcal{X}$, then $\sup_{T\in\mathcal{A}} \|T\| < \infty$.

Proof. (b) follows from (a) since a Banach space is non-meager in itself.

Proof of (a): Let

$$E_n n = \{x \in \mathcal{X} : \sup_{\mathcal{A}} \|Tx\| \leq n\} = \cap_{T\in\mathcal{A}} x \in \mathcal{X} : \|Tx\| \leq n.$$

These are closed sets so some E_n contains a non-trivial closed ball $\overline{B(x_0, r)}$. But then $\overline{B(0, r) \subset E_{2n}}$ for

$$\begin{aligned} \|x\| < r \;\Rightarrow\; x + x_0 \in B(r, x_0) &\Rightarrow \|Tx\| = \|Tx - Tx_0 + Tx_0\| \\ &\Rightarrow \|Tx\| = \|Tx - Tx_0\| + \|Tx_0\| \Rightarrow \|Tx\| = n + n, \end{aligned}$$

Hence $\|T\| \leq 2n$ for all $T \in \mathcal{A}$. $\square$

Application: Divergence of Fourier series. The Fourier series of f on $\mathbb{T}$ is the limit the partial sums

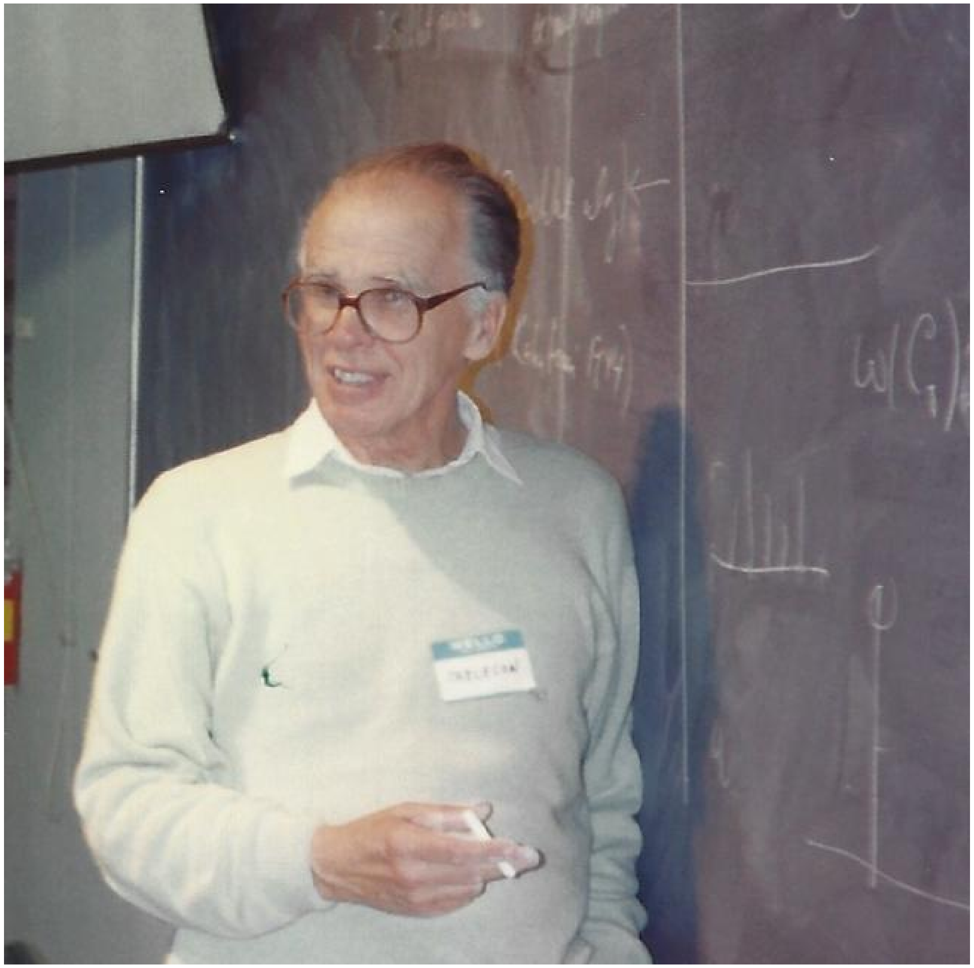
$$S_N f(t) = \sum_{k=-N}^{N} \hat{f}(n) \exp(-ikt).$$

If $S_N(x)$ converges for every continuous f and x to some limit then the Uniform Boundedness Principle says the norms of the operators $f \to S_n f$ are uniformly bounded. But we can explicitly compute the norms

$$\begin{aligned} \|S_n\| &= \|\sum_{k=-N}^{N} \exp(-ikt)\|_1 \\ &= \frac{1}{\pi}\int_0^{2\pi} \frac{|\sin(N+\frac{1}{2})t|}{t} \to \infty. \end{aligned}$$

Hence there is a continuous function whose Fourier series diverges somewhere.

In one of the high points of 20th century mathematics Lennart Carleson proved the Fourier series of a continuous function (even L^2) converges almost everywhere.



Lennart Carleson (1928–present)

5.4 Topological Vector Spaces **Defn:** A topological vector space (TVS) is a vector space $\mathcal{X}$ over a field K ($\mathbb{R}$ or $\mathbb{C}$) with a topology that makes addition and scalar multiplication continuous from $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$ or $K \times \mathcal{X} \to \mathcal{X}$ respectively.

Defn: A set A is convex if $x, y \in A$ implies $tx + (1-t)y \in A$ for all $0 \le t \le 1$.

Defn: A TVS is locally convex if there is basis of convex open sets for the topology.

A normed vector spaces is a TVS. Most other examples of TVSs are generated by families semi-norms.

Examples:

$C(\mathbb{R})$ with seminorms $\sup_{|x|\leq n} |f(x)|$.

$L^1_{loc}(\mathbb{R})$ with seminorms $\int_{-n}^{n} |f(x)|dx$

Distributions: C^∞ functions with seminorms $\|D^\alpha f\|_u$ (u = supremum norm).

Weak* topology on dual space of Banach space = weakest topology making all bounded linear functionals continuous.

Unfortunate fact: There is no norm on the space $C^\infty([0,1])$ of infinitely differentiable functions on $[0,1]$ with respect to which $\frac{d}{dx}$ is bounded.

Proof. Let $f_\lambda(x) = e^{\lambda x}$. Then

$$\frac{d}{dx} f_\lambda = \lambda f_\lambda,$$

so the operator norm of $\frac{d}{dx}$ is $\geq \lambda$. $\square$

Options:

(1) treat $\frac{d}{dx}$ as an unbounded operator

(2) treat $\frac{d}{dx}$ as a map between different spaces, e.g., $C^{k+1} \to C^k$.

(3) use seminorms $\sup |D^k f|$ on C^∞. Theory of distributions.

Theorem 5.14: Let $\{p_\alpha\}_A$ be a family of seminorms on the vector space $\mathcal{X}$. If $x \in \mathcal{X}$,$\alpha \in A$, and $\epsilon > 0$, let

$$U_{x\alpha\epsilon} = \{y \in \mathcal{X} : p_\alpha(y - x) < \epsilon\},$$

and let $\mathcal{T}$ be the topology generated by these sets.

(a) For each $x \in \mathcal{X}$, the finite intersections of the set $U_{x\alpha\epsilon}$, $\alpha \in A$, $\epsilon > 0$, form a basis of the topology at x.

(b) A net $\{x_i\}$ in this topology converges to x iff $p_\alpha(x_i - x)$ converges to 0 for all $\alpha \in A$.

(c) $(\mathcal{X}, \mathcal{T})$ is a locally convex topological vector space.

Proof. **Proof of a:** We have to show that given $x \in \mathcal{X}$ and any finite intersection of basis elements containing x (these elements may correspond to different points in $\mathcal{X}$), there is basis element for x contained in the intersection.

So if $x \in \cap_1^k U_{x_j \alpha_j \epsilon_i}$ set

$$\delta_j = \epsilon_j - p_{\alpha_j}(x - x_j) > 0.$$

Then by the triangle inequality for semi-norms

$$\cap_1^k U_{x \alpha_j \delta} \subset \cap_1^k U_{x_j \alpha_j \epsilon_i}.$$

Then apply Prop 4.4 (the topology generated by a collections of subsets of $\mathcal{X}$ consists of $\emptyset$, $\mathcal{X}$ and all unions of finite intersections of the collection).

Proof of b: By (a), it suffices to observe that $p_\alpha(x - x + i) \to 0$ iff $\{x_i\}$ is eventually in $U_{x\alpha\epsilon}$ for every $\epsilon > 0$.

Proof of c: To prove a map is continuous it suffices to show convergent nets map to convergent nets (Prop 4.19).

If $x_i \to x$ and $y_i \to y$ then

$$p_\alpha((x_i + y_i) - (x + y)) \leq p_\alpha(x_i - x) + p_\alpha(y_i - y) \to 0.$$

Thus addition is continuous. If $\lambda_i \to \lambda$, then eventually $|\lambda_i| \leq C = |\lambda| + 1$, so

$$\begin{aligned} p_\alpha(\lambda_i x_i \lambda x) &\leq p_\alpha(\lambda_i(x_i - x)) + p_\alpha((\lambda_i - \lambda)x) \\ &\leq C p_\alpha(x_i - x) + |\lambda_i - \lambda| p_\alpha(x) \end{aligned}$$

so scalar multiplication is continuous. Thus $(\mathcal{X}, \mathcal{T})$ is a TVS.

Finally, have to check $U_{x\alpha\epsilon}$ are convex.

$$\begin{aligned}
p_\alpha(x - [ty - (1-t)z]) &\leq p_\alpha(tx - ty)) + p_\alpha((1-t) + (1-t)z) \\
&\leq t\epsilon + (1-t)\epsilon \\
&\leq \epsilon \quad \square
\end{aligned}$$

Prop. 5.15: Suppose $\mathcal{X}$ and $\mathcal{Y}$ are vector spaces with topologies defined, by the families $\{p_\alpha\}_A$ and $\{q_\beta\}_B$ of seminorms and $T : \mathcal{X} \to \mathcal{Y}$ is a linear map. Then T is continuous iff for each $\beta \in B$ there exist $\{\alpha_1, \ldots, \alpha_k\} \in A$ and $C > 0$ so that $q_\beta(Tx) \leq C \sum p_{\alpha_j}(x)$.

Proof. Proof. If the bound holds, and x_i is a net converging to x then

$$p_\alpha(x_i - x) \to 0$$

for all α, so

$$q_\beta(Tx_i - Tx) \leq C \sum p_{\alpha_j}(x_i - x) \to 0.$$

Thus T maps convergent nets map to convergent nets, so it is continuous by Prop 4.19.

If T is continuous, then for every $\beta \in B$ there is neighborhood U of zero on which $q_\beta(Tx) < 1$. We can take U to be a basis element

$$U \subset \cap_1^k U_{x\alpha_j\epsilon_j}.$$

Take $\epsilon = \min \epsilon_j$. Then

$$p_{\alpha_j}(x) < \epsilon \forall j \quad \Rightarrow \quad q_\beta(Tx) < 1.$$

Suppose $p_{\alpha_j}(x) > 0$ for some j. Let

$$y = \frac{\epsilon x}{\sum p_{\alpha_j}(x)}, \text{ or } x = \frac{1}{\epsilon} \cdot y \cdot \sum p_{\alpha_j}(x).$$

Then $p_{\alpha_j}(y) < \epsilon \forall j$, so

$$q_\beta(Tx) = \sum \frac{1}{\epsilon} p_{\alpha_j}(x) q_\beta(Ty) \le \frac{1}{\epsilon} \sum p_{\alpha_j}(x).$$

On the other hand, if $p_{\alpha_j}(x) = 0 \forall j$, then $p_{\alpha_j}(rx) = 0$ for all j and all $r > 0$.

Hence

$$rq_\beta(Tx) = q_\beta(T(rx)) < 1$$

for all $r > 0$, which is only possible if it equals 0. Thus the inequality holds in this case too. $\square$

Prop. 5.16: Let $\mathcal{X}$ be a vector space equipped with the topology defined by a family $\{p_\alpha\}$ of seminorms.

(a) $\mathcal{X}$ is Hausdorff iff for each $x \neq 0$ there exists $\alpha \in A$ such that $p_\alpha(x) \neq 0$.

(b) If $\mathcal{X}$ is Hausdorff and A is countable, then $\mathcal{X}$ is metrizable with a translation invariant metric (i.e., $\rho(x, y) = \rho(x + z, y + z)$ for all $x, y, z \in \mathcal{X}$).

Folland leaves proof to reader. I will do the same

Since Hahn-Banach theorem was stated and proved for semi-norms, it applies to TVS. It guarantees the existence of lots of continuous linear functionals on a TVS $\mathcal{X}$, enough to separate points, if $\mathcal{X}$ is Hausdorff.

Defn: In a topological vector space $\mathcal{X}$, a net $\{x_i\}_I$ is called Cauchy if the net $\{(x_i - x_j\}_{I\times I}$ converges to zero.

Defn: $\mathcal{X}$ is complete if every Cauchy net converges in $\mathcal{X}$.

When $\mathcal{X}$ is first countable this is equivalent to every Cauchy sequence converging.

Defn: A Fréchet space is a complete Hausdorff topological vector space whose topology is defined by a countable family of seminorms.

Maurice Fréchet (1878–1973)

One of the most useful procedures for constructing topologies on vector spaces is by requiring the continuity of certain linear maps.

Defn: Suppose that $\mathcal{X}$ is a vector space, $\mathcal{Y}$ is a normed linear space, and $\{T_\alpha\}_A$ is a collection of linear maps from $\mathcal{X} \to \mathcal{Y}$. The weak topology $\mathcal{T}$ generated by $\{T_\alpha\}$ is the weakest topology making all these maps continuous.

$\mathcal{T}$ is the topology defined by the seminorms $p_\alpha(x) = \|T_\alpha x\|$.

The usual TVS topology on C^∞ is of this from when $T_k = \frac{d^k}{dx^k}$.

Defn: If $\mathcal{X}$ is a normed vector space the weak topology on $\mathcal{X}$ is the topology generated by $\mathcal{X}^*$, i.e., the weakest topology making $x \to y(x)$ continuous for all $y \in X^*$.

Defn: If $\{x_i\}$ is a net in $\mathcal{X}$ we say if converges weakly if $y(x_i) \to y(x)$ for all $y \in X^*$.

Example: $(L^1([0,1]))^* = L^\infty([0,1])$. A sequence in L^1 converges weakly if

$$\int f_n g dx \to \int f g dx,$$

for every $g \in L^\infty$.

$$\sin(nx) \to 0 \text{ weakly.}$$

This is the Riemann-Lebesgue Lemma.

Defn: Let $\mathcal{X}$ be a normed vector space, $\mathcal{X}*$ its dual space. The weak-star topology (or weak*-topology) is the weakest topology for which the mappings $y \to y(x)$ are continuous for all $x \in X$.

In general, weak* topology is different than the weak topology on $\mathcal{X}^*$; that one is generated by elements of $\mathcal{X}^{**}$.

Defn: Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. The topology on $L(\mathcal{X}, \mathcal{Y})$ generated by the evaluation maps $T \mapsto Tx$ for $x \in \mathcal{X}$ is called the **strong operator topology** on $L(\mathcal{X}, \mathcal{Y})$.

Defn: The topology generated by the linear functionals $T \to f(Tx)$ with $x \in \mathcal{X}$, $f \in \mathcal{X}^*$ s called the **weak operator topology** on $L(\mathcal{X}, \mathcal{Y}))$.

These topologies are best understood in terms of convergence:

$$T_\alpha \to T \text{ strongly iff } T_\alpha x \to Tx \text{ in the norm topology,}$$

$$T_\alpha \to T \text{ in weak operator topology iff } T_\alpha x \to Tx \text{ in the weak topology on} \mathcal{Y}.$$

We have:

$$\text{weak operator topology} \subset \text{strong operator topology} \subset \text{norm topology}$$

Fewer open sets means a weaker topology.

Fewer open sets means easier to be compact.

Prop. 5:17: Suppose $\{T_n\}_1^\infty \subset L(\mathcal{X},\mathcal{Y})$, $\sup_n \|T_n\| < \infty$ and $T \in L(\mathcal{X},\mathcal{Y})$. If $\|T_n x - Tx\| \to 0$ for all x in a dense subset D of $\mathcal{X}$, then $T_n \to T$ strongly.

Proof. Let $C = \sup_n \|T_n\|$. For $x \in \mathcal{X}$ and $\epsilon > 0$, choose $x' \in D$ such that $\|x - x'\| < \epsilon/3C$. Choose n large enough so that $\|Tx' - T_n x'\| < \epsilon/3$. Then $T_n x \to Tx$ in norm since

$$\begin{aligned}\|T_n x - Tx\| &\leq \|T_n x - Tx'\| + \|T_n x' - Tx'\| + \|Tx' - Tx\| \\ &\leq 2C\|x' - x\| + \epsilon/3 < \epsilon \quad \square\end{aligned}$$

Alaoglu's Theorem: If $\mathcal{X}$ is a normed vector space, the closed unit ball in $\mathcal{X}^*$ is compact in the weak* topology.

Proof. For each $x \in \mathcal{X}$ let

$$D_x = \{z \in \mathbb{C} : |z| \leq \|x\|\},$$

and let $D = \prod_{x \in X} D_x$ Then D is compact by Tychonoff's theorem. The elements of D are precisely the complex-valued functions f on $\mathcal{X}$ such that $|f(x)| \leq \|x\|$ for all $x \in \mathcal{X}$. The closed unit ball B^* of $\mathcal{X}^*$ are the linear elements of this collection.

Both the product topology and weak* topologies on B^* correspond to the topology of pointwise convergence. Since closed subsets of compact sets in a Hausdorff space are compact, it suffices to show B^* is closed, i.e., the pointwise limit of linear functions is linear. But if f_α is a net converging to f then

$$f(ax + by) = \lim_\alpha f_\alpha(ax + by) = a \lim_\alpha f_\alpha(x) + b \lim_\alpha f_\alpha(y) = af(x) + b(f(y). \quad \square$$

Example: finite signed measures on compact X are the dual space of $C(X)$ (Chapter 7). So given a sequence of probability measures on X there is a subsequence $\{\mu_n\}$ and a probability measure μ so that $\lim \int g d\mu_n = \int g\mu$ for all $g \in C(X)$.

We use this all the time in dynamics. If $f : X \to X$ we can define μ_n by putting mass $1/n$ on the first n iterates of a point x iterated. Then there is a limiting measure of a subsequence, and we can show this is f-invariant.

Patterson-Sullivan measure on limit set of Kleinian group is defined in a similar way: limit of discrete measures on a group orbit.

Many examples of proving a extremal measure for some inequality exists, e.g., minimize

$$\int\int \log \frac{1}{|x-y|} d\mu(x) d\mu(y)$$

over all probability measures μ on $[0,1]^2$.

5.5 Hilbert Spaces

Defn: An inner product on a complex vector space $\mathcal{H}$ is a map $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$ denoted $\langle x, y \rangle$ so that

(1) $\langle ax + by, z \rangle a \langle a, z \rangle + b \langle y, z \rangle$

(2) $\langle x, y \rangle \overline{\langle x, y \rangle}$.

(3) $\langle x, x \rangle \in (0, \infty)$ if $x \neq 0$.

We deduce $\langle x, ay + bz \rangle = \bar{a} \langle x, y \rangle \bar{b} \langle x, z \rangle$

Defn: A Hilbert space is a Banach space whose norm is given by a inner product $\|x\| = \langle, x, y, \rangle$.

Example: $L^2(\mu)$ is a Hilbert space, with

$$\langle f, g \rangle = \int f \overline{g} d\mu.$$

The Schwarz Inequality: $|\langle x, y\rangle| \|x\| \cdot \|y\|$ with equality iff x and y are linearly dependent.

Proof. If $\langle x, y\rangle = 0$, the result is obvious.

If $\angle x, y\rangle \neq 0$, then x, y are both non-zero. Set $\alpha = \mathrm{sgn}\langle \mathrm{x}, \mathrm{y}\rangle$ and $z = \alpha y$, so that

$$\langle x, z\rangle = \langle z, x\rangle = |\langle x, y\rangle|$$

and $\|x\| = \|y\|$. For $t \in \mathbb{R}$,

$$\langle x - tz, x - tz\rangle = \|x\|^2 - 2t|\langle x, y\rangle| + t^2 2\|y\|^2.$$

The expression on the right is a quadratic function of t whose absolute minimum occurs at $t = \angle x, y\rangle|/\|y\|^2$. Setting t equal to this value, we obtain

$$0 \leq \|x - tz\|^2 = \|x\|^2 - |\langle x, y\|^2/\|y\|^2,$$

with equality iff $x - tz = x - \alpha ty = 0$, which the desired result is immediate. $\square$

Prop. 5.20: The function $x \to \|x\|$ is a norm.

Proof. That $\|x\| = 0$ iff $x = 0$ and that $\|\lambda x\| = |\lambda| \cdot \|x\|$ are easy to check.

To prove the triangle inequality, note

$$\|x + y\|^2 = \langle x + y, x + y\rangle = \|x\| + \mathrm{Re}\langle x, y\rangle + \|y\|^2,$$

so

$$\|x + y\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2. \quad \square$$

We can rewrite

$$\|x+y\|^2 = \|x\|^+\mathrm{Re}\langle x,y\rangle + \|y^2,$$

as

$$\|x+y\|^2 - \|x\|^-\|y\|^2\mathrm{Re}\langle x,y\rangle.$$

This lets us write the inner product in terms of the norm.

The polarization identity

$$\langle x,y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2)$$

Prop. 5.21: If $x_n \to x$ and $y_n \to y$ then $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

Proof.

$$
\begin{aligned}
|\langle x_n, y_n \rangle \langle x, y \rangle| &= |\langle x_n - x, y_n \rangle + \langle x, y_n - y \rangle| \\
&\leq |\|x_n - x\| \cdot \|y_n\| + \|x\| \cdot \|y_n - y\| \\
&\to 0 \quad \square
\end{aligned}
$$

The Parallelogram law: $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

Proof. Add the equalities

$$\|x \pm y\|^2 = \|x\| \pm \mathrm{Re}\langle x, y\rangle + \|y^2. \quad \square$$

This is useful to show that a norm is not a Hilbert space norm.

Show $L^1(\mathbb{R})$ is not a Hilbert space.

Defn: x, y are orthogonal if $\langle x, y\rangle = 0$. Denoted $x \perp y$.

Defn: if $E \subset \mathcal{H}$ then $E^{\perp} = \{y : x \perp y \forall x \in E\}$. $\langle x, y\rangle = 0$.

The Pythagorean Theorem: If $x_1, \ldots, x_m \in \mathcal{H}$ are pairwise orthogonal then

$$\|\sum x_j\|^2 = \sum \|x_j\|^2.$$

Proof. By definition

$$\|\sum x_j\|^2 = \sum_{j,k} \langle x_j, x_k\rangle.$$

All the terms vanish except when $j = k$, giving $\|x_k\|^2$. $\square$

Theorem 5.34: If $\mathcal{M}$ is closed subspace of $\mathcal{H}$ then $H = M \oplus M^{\perp}$. In other words, every vector $x = y + z$ in $\mathcal{H}$ can be expressed uniquely as a sum of $y \in M$ and $z \in M^{\perp}$. These are the unique elements of these spaces that are closest to x.

Proof. For $x \in \mathcal{H}$ define

$$\delta = \inf\{\|x - y\} : y \in \mathcal{M}\}.$$

Let $\{y_n\}$ be a sequence converging to the infimum. By the parallelogram law

$$2(\|y_n - x\|^2 + \|y_m - x\|^2) = \|y_n - y_m\|^2 + \|y_n + y_m 2 - x\|^2$$

Since $\frac{1}{2}(y_n + y_m) \in \mathcal{M}$

$$\begin{aligned}
\|y_n - y_m\|^2 &= 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4\|\frac{1}{2}(y_m + y_n) - x\|^2 \\
&= 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4\delta^2 \\
&\to 2\delta^2 + 2\delta^2 - 4\delta^2 = 0
\end{aligned}$$

Thus $\{y_n\}$ is Cauchy sequence and converges to some $y \in \mathcal{M}$ since $\mathcal{M}$ is closed. Moreover, $\|x - y\| = \delta$.

Let $z = x - y$. We claim $z \in \mathcal{M}^\perp$. For any $u \in \mathcal{M}$ multiply by a scalar so that inner product with x is real. Then

$$f(t) = \|z + tu\|^2 = \|z\|^2 + 2t\langle z, u\rangle + t^2\|u\|^2$$

is real quadratic in t with minimum δ^2 at $t = 0$, and so has derivative zero at $t = 0$. Hence $\langle z, u\rangle 0$, so $z \in \mathcal{M}^\perp$

If $z' \in \mathcal{M}^\perp$ then $x - z \perp z' - z$ so t

$$\|z' - x\|^2 = \|z - x\|^2 + \|z - z'\|^2$$

so z is unique closest point to x. Same argument shows y is closest point of M to x

If $x = y' + z'$ in the same spaces, then $y - y' = z - z' \in \mathcal{M} \cap \mathcal{M}^\perp = \{0\}$. Thus the sum is unique. $\square$

If $y \in \mathcal{H}$ then $x \to \langle x, y \rangle$ is a bounded linear functional of norm $\|y\|$. Thus $\mathcal{H}$ embeds in its dual space. In fact, this embedding is onto.

Theorem 5.25: If $f \in \mathcal{H}^*$ then there is a unique $y \in \mathcal{H}$ so that $f(x) = \langle x, y \rangle$ for all $x \in \mathcal{H}$.

Proof. To prove uniqueness suppose

$$\langle x, y \rangle = \langle x, y' \rangle$$

for all $x \in \mathcal{H}$. Then

$$\langle x, y - y' \rangle = 0$$

for all $x \in \mathcal{H}$. Taking $x = y - y'$ gives a contradiction unless $y = y'$.

If f is the zero functional take $y = 0$. Otherwise let $\mathcal{M} = \{x \in \mathcal{H}; f(x) = 0\}$

If f is the zero functional take $y = 0$. Otherwise let $\mathcal{M} = \{x \in \mathcal{H}; f(x) = 0\}$
Then $\mathcal{M}$ is closed and non-empty so we can choose $z \in M \perp$ with norm 1. Set $u = f(x)z - f(z)x$. Then

$$f(u) = f(x)f(z) - f(x)f(z) = 0$$

so $u \in \mathcal{M}$ so

$$0 = \langle u, z \rangle = f(x)\|z\|^2 - f(z)\langle x, z \rangle = f(x) - \langle x, \overline{f(z)}z \rangle.$$

Hence

$$f(x) = \langle x, \overline{f(z)}z \rangle,$$

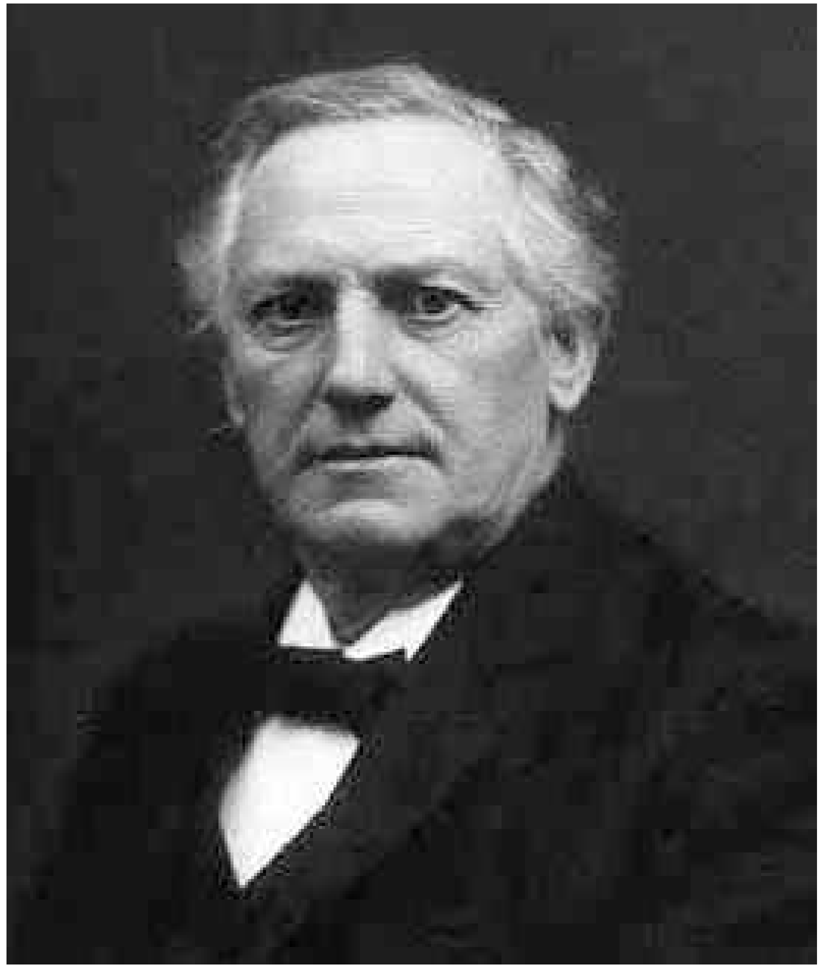
so we can take $y = \overline{f(z)}z$. $\square$

Defn: A subset $\{u_\alpha\}$ is orthonormal if every element has unit length and pair are perpendicular.

Gram-Schmidt: given a sequence $\{x_n\}$ of linearly independent vectors in $\mathcal{H}$ we can convert it to an orthonormal set by setting

$$y_n = x_n - \sum_{k=1}^{n-1} \langle x, x_k \rangle x_k,$$

$$u_n = y_n / \|y_n\|.$$



Jorgen Pedersen Gram (1850–1916)

Erhart Schmidt (1876–1959)

Bessel's inequality: If $\{u_\alpha\}$ is orthonormal then for any $x \in \mathcal{H}$

$$\sum_{\alpha \in A} |\langle x, u_\alpha \rangle|^2 \leq \|x\|^2.$$

Proof. Enough to prove this for finite subsets F of A.

$$\begin{aligned} 0 &\leq \|x - \sum_{\alpha \in F} \langle x, u_\alpha \rangle u_\alpha\|^2 \\ &\leq \|x\|^2 - 2\mathrm{Re}\langle x, \sum_{\alpha \in F} u_\alpha \rangle u_\alpha \rangle + \|\sum_{\alpha \in F} \angle x, u_\alpha \rangle u_\alpha\|^2 \\ &\leq \|x\|^2 - 2\sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2 + \sum_{\alpha \in F} |\angle x, u_\alpha \rangle|^2 \\ &\leq \|x\|^2 - \sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2 \end{aligned}$$

$\square$

This uses

$$\begin{aligned}\langle x, \sum_{\alpha\in F} u_\alpha\rangle u_\alpha\rangle &= \sum_\alpha \overline{\langle x, u_\alpha\rangle}\langle x, u_\alpha\rangle \\ &= \sum_\alpha |\langle x, u_\alpha\rangle|^2\end{aligned}$$

Theorem: If $\{u_\alpha\}$ is an orthonormal set TFAE:

(a) (Completeness) $\{u_\alpha\}^\perp = \{0\}$

(b) (Parseval's Identity) $\|x\|^2 = \sum_\alpha |\angle x, u_\alpha\rangle|^2$ for all x.

(c) For all x, $x = \sum_\alpha \langle x, u_\alpha\rangle u_\alpha$, where the sum has only countably many non-zero terms and converges in norm topology no matter how the terms are ordered.

Friedrich Wilhelm Bessel (1784–1846)

(1784–1846)

Proof of (a) $\Rightarrow$ (c): Let $\alpha_1, \alpha_2, \ldots$ be any enumeration of the α's that have non-zero inner product with x. By Bessel's inequality

$$\sum |\langle x, u_\alpha \rangle|^2 < \infty$$

so by the Pythagorean theorem

$$\| \sum_n^m \langle x, u_\alpha \rangle u_\alpha \|^2 = \sum_n^m |\langle x, u_\alpha|^2 \to 0$$

as $n, m \nearrow \infty$. Thus the series is Cauchy and hence converges since $\mathcal{H}$ is complete. Set

$$y = x - \sum \langle x, u_\alpha \rangle u_\alpha .$$

Then $y \perp u_\alpha$ for all α, so $y = 0$ by (a).

Proof of (c) $\Rightarrow$ (b): We have

$$\|x\|^2 - \sum_\alpha |\angle x, u_\alpha \rangle|^2 = \|x - \sum_\alpha \angle x, u_\alpha \rangle\|^2 \to 0$$

Proof of (b) $\Rightarrow$ (a): If $x \perp u_\alpha$ for all α then

$$\|x\|^2 = \sum_\alpha |\langle x, u_\alpha \rangle|^2 = \sum 0 = 0$$

Defn: A set satisfying the conclusions of Theorem 5.27 is called an orthonormal basis.

Example: let $e_n(n) = 1$ and $e_n(k) = 0$ otherwise. Then $\{e_n\}$ is an orthogonal basis of ℓ_2.

Example: $\exp(inx)$ is an orthogonal basis of $L^2([0, 2\pi])$.

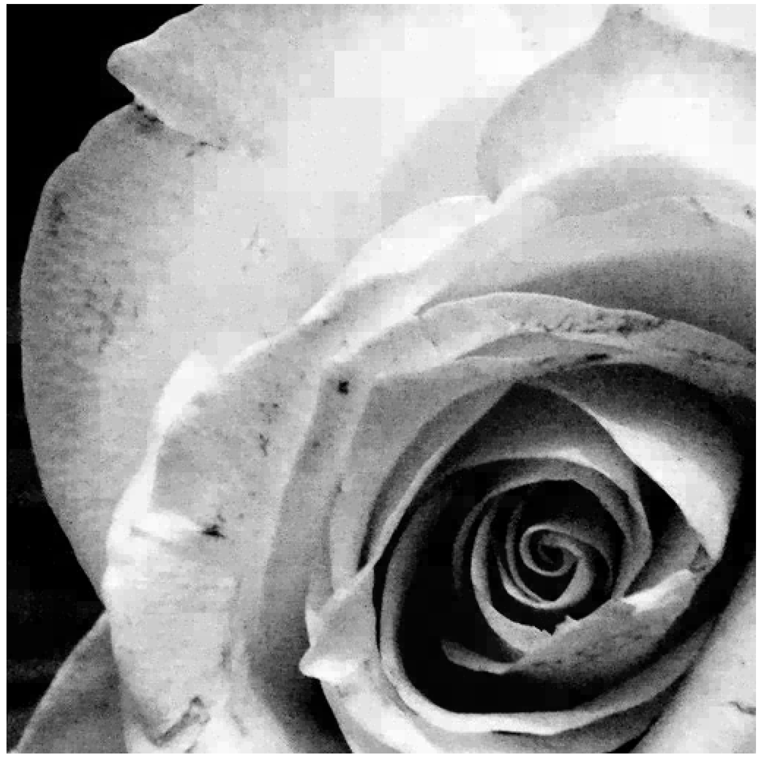
Example: z^n, $n \geq 0$ is basis of $H^2(\mathbb{D})$.

Example: $\chi_x(y) = 1$ iff $y = x$ is uncountable basis of L^2 for counting measure on $\mathbb{R}$.

Example: Haar basis of $L^2([0,])$, where $h_n(x) = \mathrm{sgn}(\sin(2^{\mathrm{n}}\pi \mathrm{x})$. This is simplest wavelet basis.

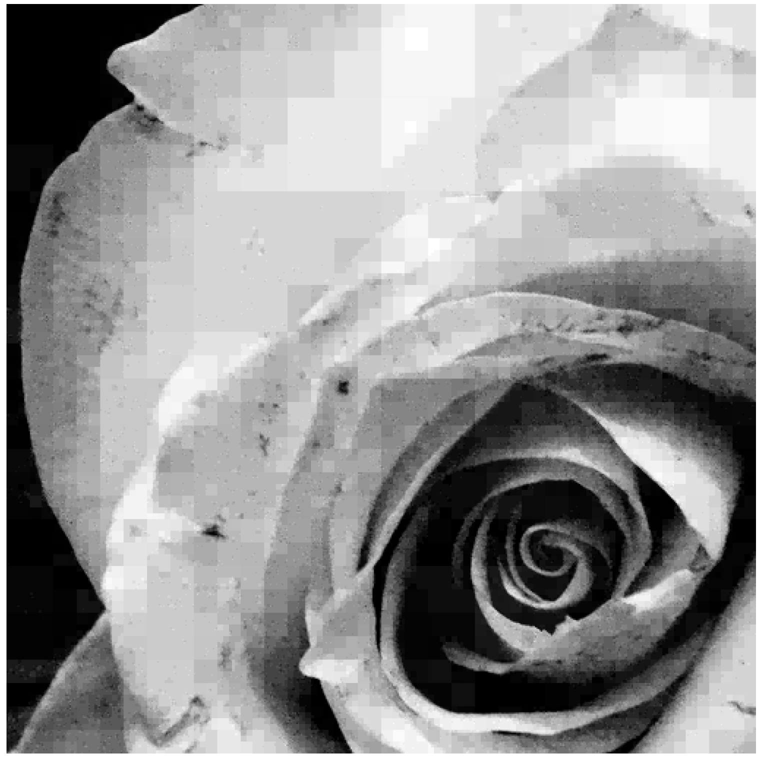
A vector on $\mathbb{R}^{349526}$

Represented using 349526 Haar functions



A vector on $\mathbb{R}^{349526}$

Represented using 105739 Haar functions

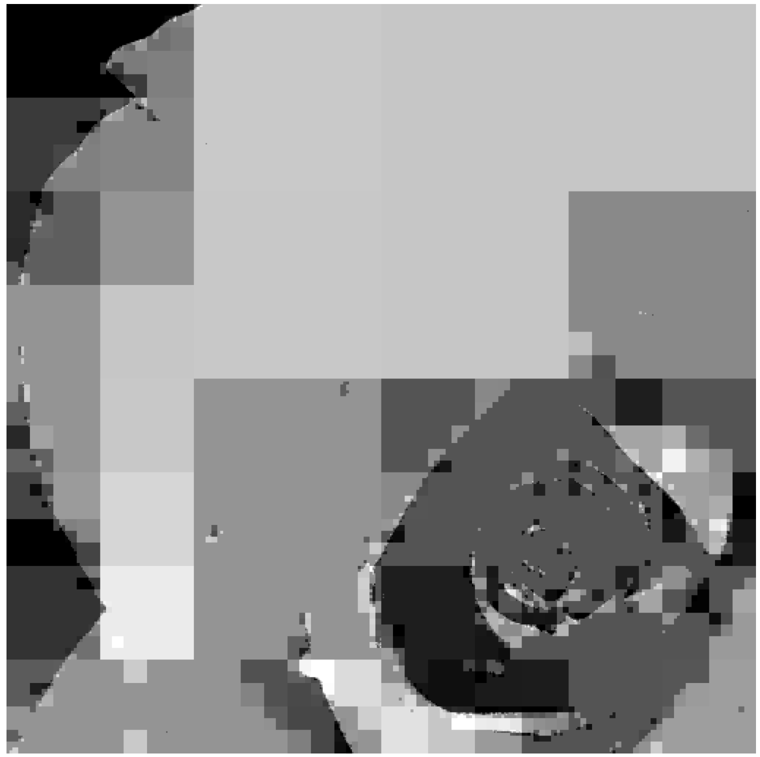


A vector on $\mathbb{R}^{349526}$

Represented using 41000 Haar functions

A vector on $\mathbb{R}^{349526}$

Represented using 11525 Haar functions



A vector on $\mathbb{R}^{349526}$

Represented using 1575 Haar functions

An orthonormal basis in a Hilbert space is an example of an unconditional, monotone basis.

Per Enflo proved a general Banach space need not have a basis at all, much less an unconditional basis.

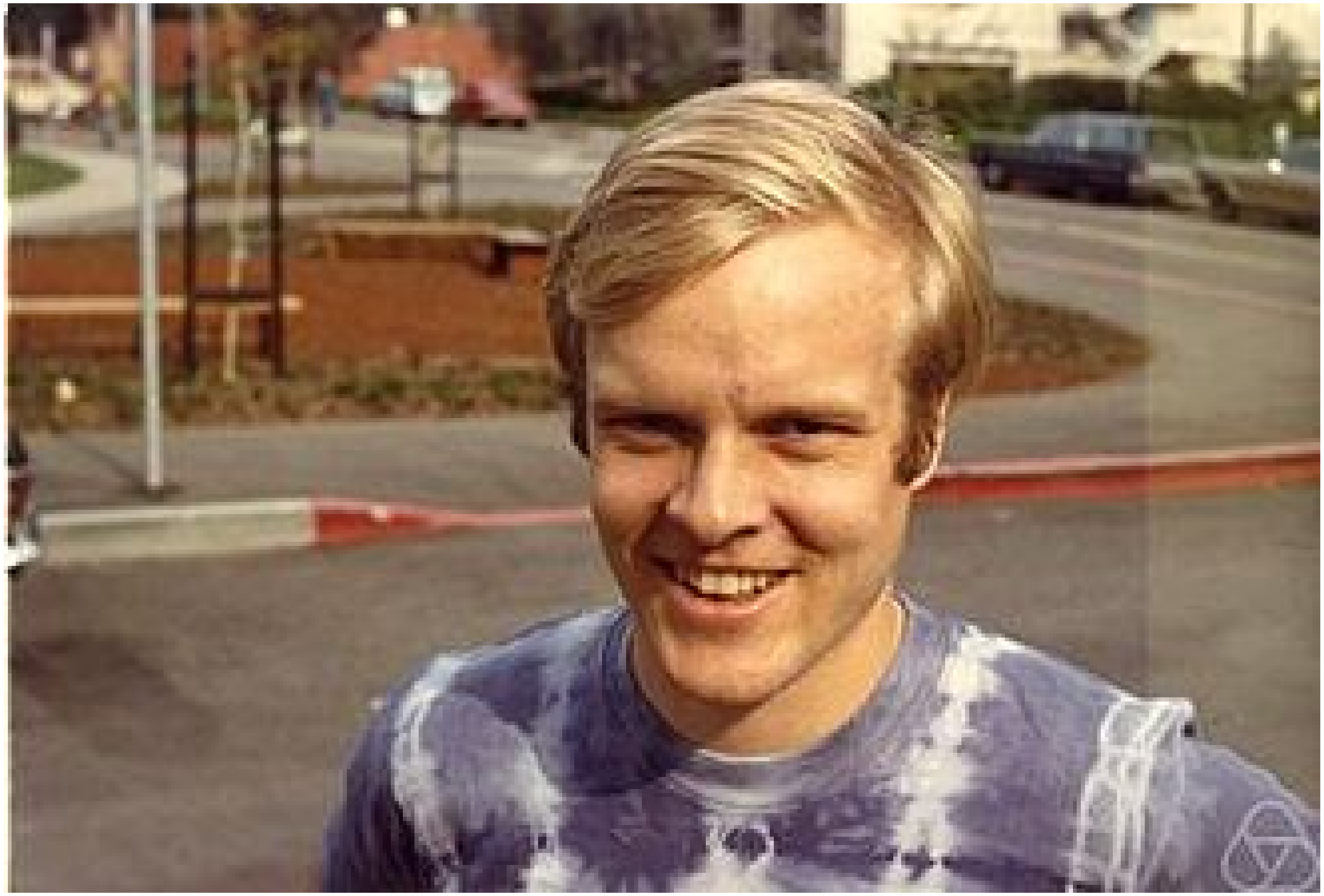
It is now known that the space of bounded operators from ℓ^2 to itself does not have the approximation property.

Grothendieck had reduced finding a space with no basis to finding a space without the approximation property: every compact operator is a limit of finite rank operators.

Defn: an operator is *compact* if the image of any bounded set is precompact.

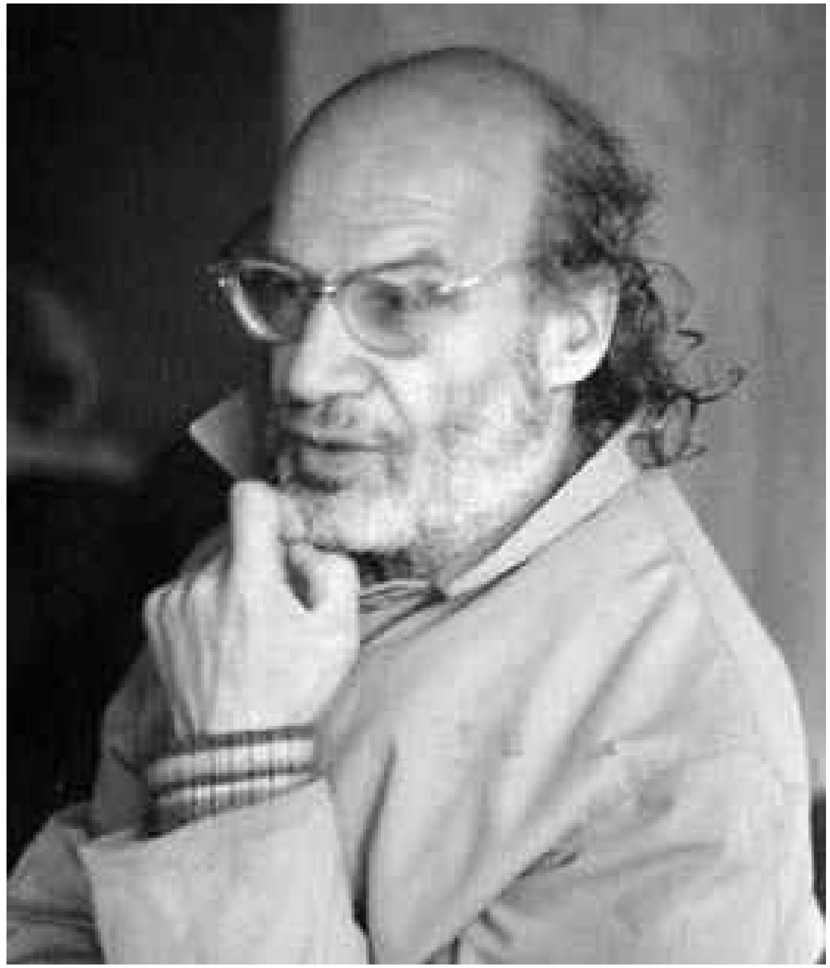
Defn: an operator is *finite rank* it is image is finite dimensional.

Every limit of finite rank operators is compact. In a Hilbert space, the converse is true. (Enflo showed false in Banach spaces.)



Per Enflo (1944–present)

Stanislaw Mazur presenting Per Enflo with a live goose on Polish TV in 1972 for solving Problem 153 in the Scottish Book.



Alexander Grothendieck (1928–2014)

Prop 5.28: Every Hilbert space has an orthonormal basis.

Proof. Zorn's lemma shows that the collection of orthonormal sets is partially ordered and has a maximal element. Maximality means the collection is orthogonal to everything, which is (a) in Theorem 5.27. □

Prop. 5.29: A Hilbert space is separable iff it has a countable orthonormal basis. in which case every orthonormal basis is countable.

Proof. If $\{x_n\}$ is a countable dense set in $\mathcal{H}$, then by discarding recursively any x_n that are in the linear span of earlier ones, we obtain a linearly independent sequence $\{y_n\}$ whose linear span is dense. Application of the Gram-Schmidt process yields an orthonormal sequence $\{u_n\}$ whose linear span is dense $\mathcal{H}$ and which is therefore a basis.

Conversely, if $\{u_n\}$ is a countable orthonormal basis, the finite linear combinations of the u_n with coefficients in a countable dense subset of $\mathbb{C}$ form a countable dense set. Moreover, if v_α is another orthonormal basis, for each n the set $A_n = \{a \in A : \langle u_n, v_\alpha \rangle \neq 0\}$ is countable. By completeness of $\{u_n\}$, $A = \cup_n A_n$ so A is countable. $\square$

Defn: An invertible linear map between Hilbert spaces is **unitary** if it preserves the inner product.

Unitary maps preserve norms, so are isometries.

Prop 5.30: Let $\{u_\alpha\}_A$ be an orthonormal basis for $\mathcal{X}$. Then $x \to \langle x, u_\alpha \rangle$ is a unitary map from $\mathcal{X}$ to $\ell^2(A)$.

Proof. The map is linear, and it is an isometry by the Parseval identity. To show it is onto, let $f \in \ell^2(A)$. Then the Pythagorean theorem shows that the partial sums of the series $\sum f(\alpha)u_\alpha$ are Cauchy, so converge to some x that maps to f. $\square$

Spectrum and Spectral theory - brief summary:

Defn: A bounded operator on a Hilbert space is **symmetric** if $\langle Mx, y\rangle = \langle x, My\rangle$.

Defn: The **spectrum** σ of a bounded operator on a Hilbert space is the set of complex values λ so that $\lambda I - M$ is not invertible.

Example: if M has an eigenvalue $Mv = \lambda v$ then $(\lambda I - M)v = 0$ so this operator is not invertible. So λ is in the spectrum.

Thm 31.2 Lax: if M bounded and symmetric, the spectrum is real.

Thm 31.9 Lax: if M bounded and symmetric, then there is projection-valued measure μ on the spectrum σ so that

$$M = \int_\sigma \lambda d\mu(\lambda)$$

and, more generally,

$$f(M) = \int_\sigma f(\lambda) d\mu(\lambda)$$

for any continuous function f on σ.

The spectral measure can have atoms, a part singular to Lebesgue measure and a part absolutely continuous with respect to Lebesgue measure.

Thm 28.3 Lax: if M is compact and symmetric, then there is an orthonormal basis of eigenvectors

$$Mz_n = \alpha_n z_n.$$

The eigenvalues are real and the only accumulate at zero. This generalizes diagonalization of real symmetric matrices.

Thm 30.1 Lax: if T is compact, it can be factored as $T = UA$ where U is unitary $(UU^* = I)$ is positive, symmetric.

The eigenvalues of A are called the singular values of T.

T is called **trace class** if $\sum s_j(T) < \infty$. This sum defines a norm on trace class operators. For a trace class operator

$$\sum \langle Tz_n, z_n \rangle = \sum \lambda_k(T)$$

where $\{\lambda_k\}$ are the eigenvalues and $\{z_n\}$ is any orthonormal basis.

For a trace class operator T, we can define a **determinant** of $I + T$

$$\det(I + T) = \prod (1 + \lambda_k).$$

Defn: A operator $T$is **Hilbert-Schmidt class** if there is an orthonormal basis $\{e_k\}$ so that

$$\sum \| Te_k{}^2 < \infty.$$

This occurs iff $\sum s_k^2(T) < \infty$. Every trace class operator is the product of two Hilbert-Schmidt operators (and conversely).

David Hilbert (1862–1943)