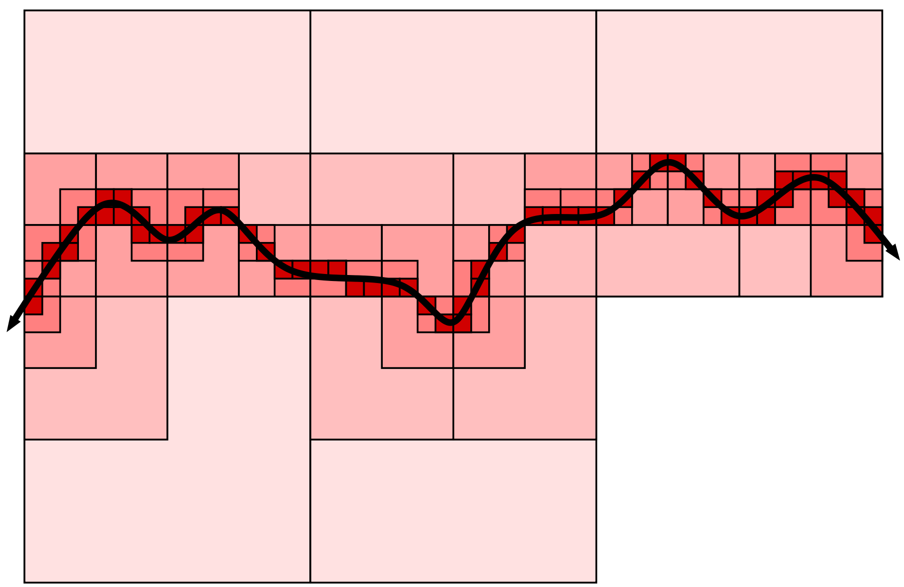
MAT 533, FALL 2021, Stony Brook University

REAL ANALYSIS II

Christopher Bishop



1. Monday, February 1, 2021

MAT 533 is a continuation of MAT 532 in the Fall of 2020.

In the Fall, Prof. Cheng covered most of Chapters 1, 2, 3 and 5. He did not cover the section on interpolation of L^p spaces. I am willing to discuss it briefly, but I am not planning to cover it in detail.

We will try to cover as much of Chapters 4, 5, 7, 8, 9, 10 as we can (not everything, though some chapters are short).

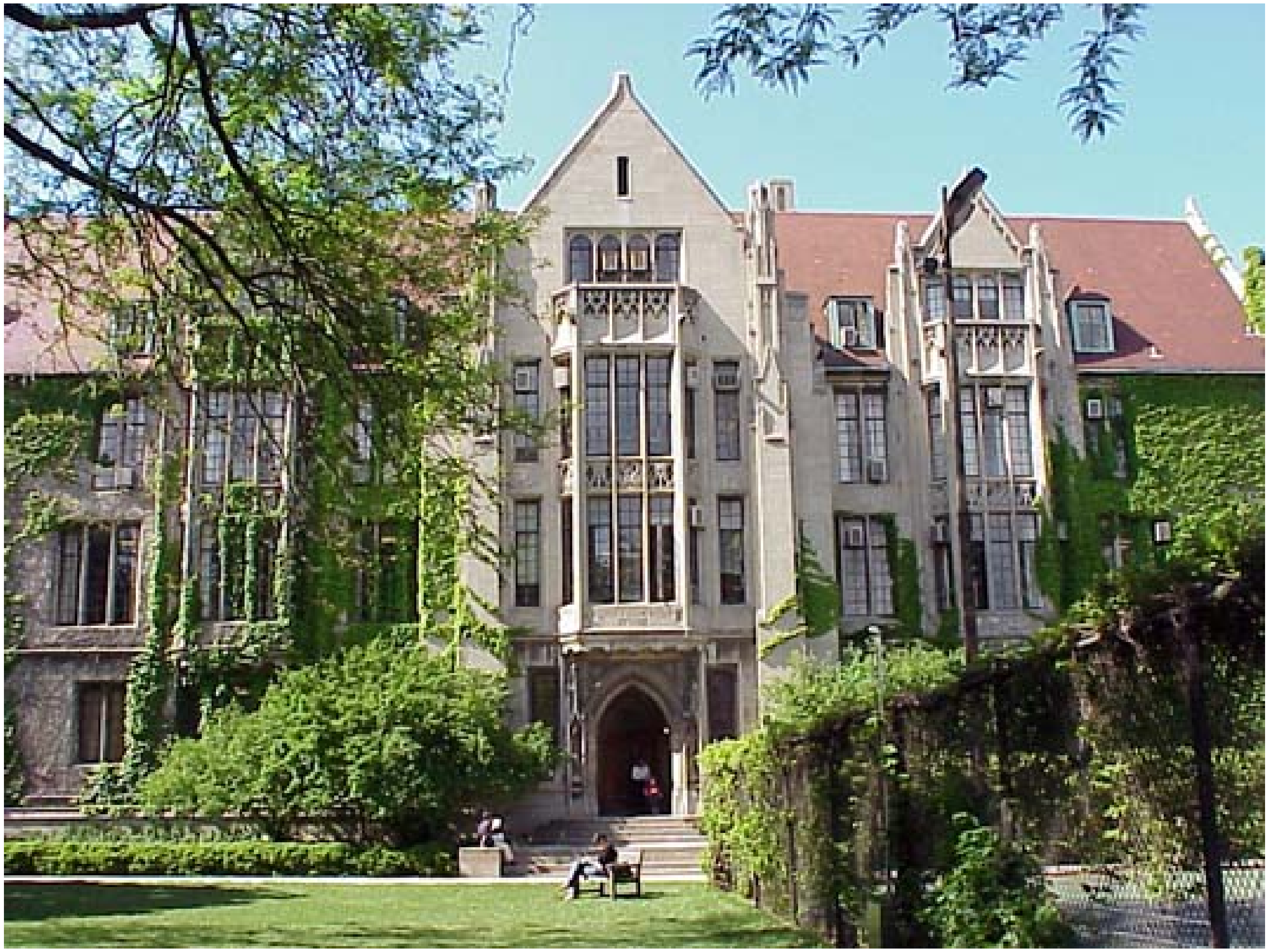
I work in analysis: conformal and quasiconformal mappings, holomorphic dynamics (especially transcendental dynamics), hyperbolic geometry, algebras generated by harmonic functions, analysis of fractals, computational geometry, optimal meshing.

Math Dept, Michigan State

Churchill College, Cambridge

New Math Dept, Cambridge

Old Math Dept, Cambridge



Math Dept, Eckhart Hall, U Chicago

Math Dept, Leet Oliver Memorial, Yale

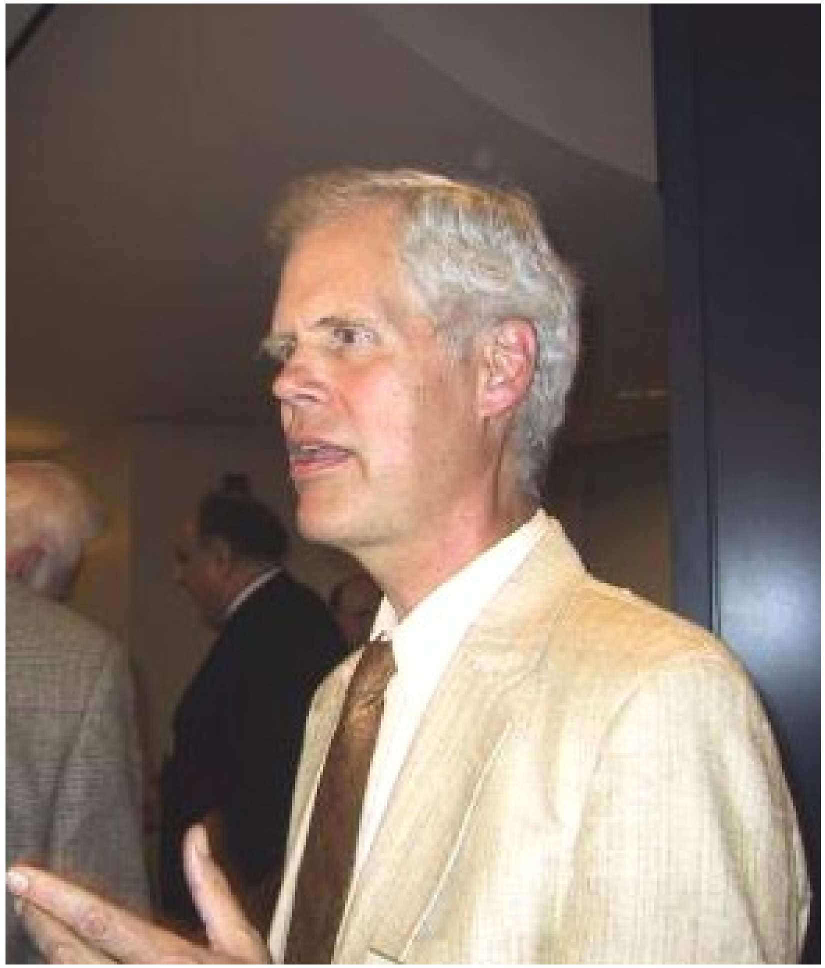
MSRI, Berkeley

Math Dept, UCLA


MATHEMATICS
PHYSICS
UMBILIC TORUS

Textbook: Real Analysis: modern techniques and their applications, 2nd Edition.

by Gerald Folland (professor emeritus at Univ. Washington, Seattle)



Gerald Folland (1947–present)

Interview with Gerald Folland

Chapter 4: Point Set Topology

4.1 Topological Spaces

4.2 Continuous Maps

4.3 Nets

4.4 Compact Spaces

4.5 Locally Compact Hausdorff Spaces

4.6 Two Compactness Theorems

4.7 The Stone- Weierstrass Theorem

4.8 Embeddings in Cubes

Chapter 5: Elements of Functional Analysis

5.1 Normed Vector Spaces

5.2 Linear Functionals

5.3 The Baire Category Theorem and its Consequences

5.4 Topological Vector Spaces

5.5 Hilbert Spaces

5.6 Notes and References

Chapter 7: Radon Measures

7.1 Positive Linear Functionals on $C_c(X)$

7.2 Regularity and Approximation Theorems

7.3 The Dual of $C_0(X)$.

7.4 Products of Radon Measures

Chapter 8: Elements of Fourier Analysis

8.1 Preliminaries

8.2 Convolutions

8.3 The Fourier Transform

8.4 Summation of Fourier Integrals and Series

8.5 Pointwise Convergence of Fourier Series

8.6 Fourier Analysis of Measures

8.7 Applications to Partial Differential Equations

Chapter 9: Elements of Distribution Theory

9.1 Distributions

9.2 Compactly Supported, Tempered, and Periodic Distributions

9.3 Sobolev Spaces

Existence theorem for ODE:

We will follow Prof. Varolin's notes (posted on webpage).

Chapter 10: Topics in Probability Theory

10.1 Basic Concepts

10.2 The Law of Large Numbers

10.3 The Central Limit Theorem

10.4 Construction of Sample Spaces

10.5 The Wiener Process

Chapter 4.3: Nets

Basic idea: nets are like sequences, but only partially ordered.

A set in a metric space is compact iff every sequence has a convergent subsequence.

A set in a topological space is compact iff every net has a convergent sub-net.

Every sequence is an net, but not conversely. It is possible for a sequence to have a sub-net that is not a sequence.

We will use nets to prove Tychonoff's theorem and in Chapter 5 when discussing weak and weak* topologies on Banach spaces, and later when discussing topologies on spaces of distributions. These are not always metrizable (but one of the most important, the weak* topology on probability measures is metrizable).

In a general topological space taking limits of all sequences in a set need not give the whole closure.

$\mathbb{C}^{\mathbb{R}}$ = all complex functions with product topology.

A sequence converges in this topology iff it converges pointwise.

$C(\mathbb{R})$ = continuous complex functions.

Theorem 2.9 says pointwise limits of continuous function are Borel. Thus "sequential closure" of $C(\mathbb{R})$ are Borel functions.

But $C(\mathbb{R})$ is dense in $\mathbb{C}^{\mathbb{R}}$ in product topology. Given any function, a neighborhood consists of all functions approximating it to within ϵ on a finite set. A continuous function can be chosen to agree with any function on a finite set.

A directed set is a set A equipped with a binary relation $\lesssim$ such that

- $a \lesssim a$ for all $a \in A$;
- if $\alpha \lesssim \beta$ and $\beta \lesssim \gamma$ then $\alpha \lesssim \gamma$;
- for any $\alpha, \beta \in S$ there is a γ with $\alpha \lesssim \gamma$; and $\beta \lesssim \gamma$.

Defn: A net in X is a mapping from a directed set A into X.

Examples:

(1) $\mathbb{N} = \{1, 2, 3, \dots\}$ with usual ordering (sequences)

(2) finite sets under inclusion. Gives nets in product topology on $\mathbb{C}^{\mathbb{R}}$.

(3) The set of neighborhoods of a point ordered by reverse inclusion.

(4) Finite partitions of an interval ordered by maximum gap length. Gives limits for defining Riemann integrals.

(5) $\mathbb{R} \setminus \{a\}$ ordered by decreasing distance to a. Defines convergence to a point.

(6) Product of directed sets. Define $(\alpha, \beta) \lesssim (\gamma, \delta)$ iff $\alpha \lesssim \gamma$ and $\beta \lesssim \delta$.

Defn: A net is $\{x_\alpha\}$ is **eventually** in $E \subset X$ if $\alpha \precsim \beta \Rightarrow x_\beta \in E$.

Defn: A net is $\{x_\alpha\}$ is **frequently** in $E \subset X$ if for all α there is a $\alpha \precsim \beta$ with $x_\beta \in E$.

Defn: A net **converges** to a point if it is eventually in every neighborhood of that point.

Defn: A net **clusters** or **accumulates** at a point if it is frequently in every neighborhood of that point.

Defn: x is an accumulation point of E if every neighborhood of x contains a point of E other than x itself.

Prop 4.18.A: x is in accumulation point of E iff there is a net in $E \setminus \{x\}$ converging to x.

Proof. If x is an accumulation point every neighborhood hits $E \setminus \{x\}$. Define a net by taking one such point from each neighborhood ordered by reverse inclusion. This net converges to x.

If x_α is a net in $E \setminus \{x\}$ converging to x then every neighborhood of x has contains some $x_\alpha \in E \setminus \{x\}$, so x is an accumulation point. $\square$

Prop 4.18.B: z is in the closure of E iff there is a net in S converging to z.

Proof. Closure of E is the union of E and its accumulation set (Prop 4.1). $\square$

Prop 4.19: A map $f : X \to Y$ of topological spaces is continuous iff every convergent net maps to a convergent net.

Proof. If f is continuous and V is a neighborhood of $f(x)$ then $U = f^{-1}(V)$ is a neighborhood of of x, so it eventually contains any net converging to x. Thus V eventually contains the image of any such net.

Conversely, if f is not continuous at x, there is a neighborhood V of $f(x)$ whose preimage U is not a neighborhood of x, so x is in the closure of the complement of U. By Prop. 4.18.B there is net in the complement converging to x, but the image is not in V, so does not converge to $f(x)$. $\square$

Defn: $\{y_\beta\}_B$ is a **subnet** of $\{x_\alpha\}_A$ if there is a map $B \to A$ so that

- $y_\beta = x_{\alpha(\beta)}$
- for each α_0 there is a β_0 so that $\beta \gtrsim \beta_0 \Rightarrow \alpha(\beta) \gtrsim \alpha_0$.

A subnet can be larger than a net "containing" it.

Consider a dense sequence in $C([0,1])$. Set of subsequences of sequence has cardinality c of $\mathbb{R}$.

But this sequence has subnets that converge to any element of $\mathbb{C}^{[0,1]}$. This space has strictly more than continuum of elements. So there are subnets that are not sequences (most of them).

Prop 4.20: x is a cluster point of X iff every net clustering at x has a subnet converging to x.

Proof. Part 1:

Suppose there is a convergent subnet y_β to x. For any neighborhood V of x, there is a a β_1 so $\beta \gtrsim \beta_1 \Rightarrow y_\beta \in V$.

By the defn of subnet, For any α choose β_2 so that $\beta \gtrsim \beta_2 \Rightarrow \alpha(\beta) \gtrsim \alpha$.

By the defn of partial order there is a $\beta_3 \gtrsim \beta_1, \beta_2$ so that $\beta \gtrsim \beta_3 \Rightarrow \alpha(\beta) \gtrsim \alpha$ and $x_\alpha \in V$. So x_α is in V frequently.

Part 2:

If x is a cluster point of $\{x_\alpha\}_A$, let $\mathcal{N}$ be the set of neighborhoods of x (ordered by inverse inclusion) and take usual partial order on $A \times \mathcal{N}$. for each (U, γ) in product choose $\alpha \gtrsim \gamma$ so $x_\alpha \in U$. This subnet converges to x. $\square$

Section 4.4 Compact Spaces:

Defn: a collection of open sets such that $X = \cup_\alpha U_\alpha$ is called an **open cover** of X.

Defn: X is **compact** if every open cover contains a finite subcover.

Defn: X is **precompact** if its closure is compact.

Defn: A family of sets X_α has the **finite intersection property** if any finite number of them has a point in common.

Prop. 4.21 X is compact if ever family of compact sets with the finite intersection property has non-empty (infinite) intersection.

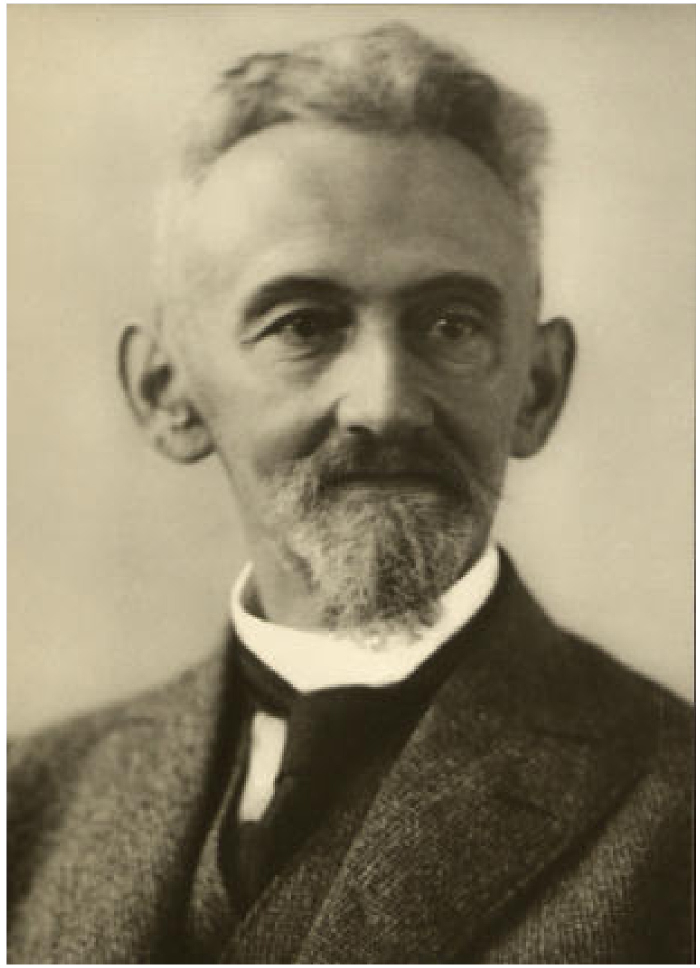
Prop. 4.22 A closed subset of a compact set is compact.

Prop. 4.23 If F is a compact subset of a Hausdorff space X and $x \notin F$, then there are disjoint open sets U, V such that $x \in U$ and $F \subset V$

Defn: A space is **Hausdorff**, a.k.a. T_2, if given $x \neq y$ there are disjoint open sets U, V with $x \in U$, $y \in V$.

Prop. 4.24 Every compact subset of a Hausdorff space is closed.

In a non-Hausdorff space, compact sets need not be closed, and intersections of compact sets need not be compact (Exercise 37). All our spaces will be Hausdorff.



Felix Hausdorff 1868–1942

Defn: A space is T_1, if given $x \neq y$ there is an open sets U with either $x \in U, y \in U^c$, or $x \in U^c, y \in U$.

Defn: A space is **normal**, a.k.a. T_4, if it is T_1 and given disjoint closed sets A, B there are disjoint open sets U, V with $A \subset U$, $B \subset V$.

Prop. 4.25: Every compact Hausdorff space is normal.

Prop. 4.26: If X is compact and $f : X \to Y$ is continuous, then $f(X)$ is compact.

Cor. 4.26: Complex valued continuous functions on a compact space are bounded.

Prop. 4.28: If X is compact and Y is Hausdorff, then a continuous 1-1 map $f : X \to Y$ has a continuous inverse, i.e., is a homeomorphism.

Thm. 4.29: If X is a topological space, TFAE:

(a) X is compact.

(b) Every net in X has a cluster point.

(c) Every net in X has a convergent subnet.

Proof. (b) $\Leftrightarrow$ (c) is Prop. 4:20.

Assume (a) holds (X is compact). If x_α is a net, let $E_\alpha = \{x_\beta : \beta \gtrsim \alpha\}$ definition of partially ordered set implies this has finite intersection property, hence so does family of closures $\overline{E_\alpha}$. By compactness, this family has non-empty intersection E.

If $x \in E$ then any neighborhood U of x hits every $E\alpha$, hence X_α clusters at x.

Assume (a) fails (X is not compact). Let $\{U_\beta\}_B$ be an open cover with no finite subcover. Let $\mathcal{A}$ be collection of finite subsets α of B ordered by inclusion and define net on A by choosing x_α not in $\cup_{\beta\in\alpha} U_\beta$. For every $x \in$, this net is eventually outside an open set containing x, so does not cluster at x. Hence it clusters nowhere. $\square$

Section 4.6: Two compactness theorems:

Defn: If $\{X_\alpha\}A$ is a family topological spaces, then $X = \prod_\alpha X_\alpha$ is the space of all maps $A \to \cup_\alpha X_\alpha$ that takes α into x_α.

Defn: The product topology is the weakest making the projection onto each coordinate continuous. The open sets are all finite intersections of inverse images of open sets under the coordinates.

Corresponds to pointwise convergence on each coordinate.

Defn: We say $p \in \prod_C X_\alpha$ is an extension of $q \in \prod_B X_\alpha$ if $B \subset C$ and $p = q$ on B.

Tychonoff's Theorem: If $\{X_\alpha\}$ is a family of compact topological spaces, then $X = \prod_\alpha X_\alpha$ is compact.

Andrei Nikolaevich Tikhonov 1906–1993

Tychonoff's Theorem: If $\{X_\alpha\}$ is a family of compact topological spaces, then $X = \prod_\alpha X_\alpha$ is compact.

Proof. We have to show any net x_i has a cluster point. Let

$$\mathcal{P} = \cup_{B \subset A} \{p \in \prod_{\alpha \in} B X_\alpha : p \text{ is a cluster point of } \pi_B(x_i)\}$$

When $B = \{\alpha\}$ is a single point, the projected net has cluster point since X_α is compact. Order $\mathcal{P}$ by inclusion.

Given a linearly ordered subset of $\mathcal{P}$, we claim the union $p^* \in \cup_{B^*} X_\alpha$ is an element of $\mathcal{P}$. Indeed, p^* has a neighborhood of the the form $U = \prod U_\alpha$ where $U_\alpha = X_\alpha$ except for finitely many indices, all of which belong to some B_l in the linear ordering. Thus the projection of x_i into $\prod_{B_l} X_\alpha$ is in $\prod_{B_l} U_\alpha$ frequently, so the projection on B^* is in U frequently, so p^* is a cluster point of it, so $p^* \in \mathcal{P}$.

By Zorn's lemma $\mathcal{P}$ has a maximal element $\overline{p} \in \prod_{\overline{B}} X_\alpha$. If $\overline{B}$ is not all of A, choose $\gamma \in A \setminus \overline{B}$. By the definition of $\mathcal{P}$ there is a net in the product over $\overline{B}$ that clusters at $\overline{p}$ and hence a subset that converges to it. Since X_γ is compact, there is a subnet that also converges when projected to X_γ, and this defines an extension o $\overline{B} \cup \gamma$, contradicting maximality. Thus $\overline{B} = A$.

Thus any net x_i has a cluster point in $\prod_A X_\alpha$, proving the product space is compact. $\square$

Max Zorn (1906–1993)

Defn: a family of functions is equicontinuous at x if for every $\epsilon > 0$ there is a neighborhood U so that $x, y \in U \Rightarrow |f(x) - f(y)| < \epsilon$.

Defn: a family of functions is equicontinuous on X if it is equicontinuous at $x \in X$.

$\{x^n\}$ is not equicontinuous on $[0, 1]$. Neither is $\{\sin(nx)\}$.

Defn: a family of functions $\mathcal{F}$ is pointwise bounded equicontinuous on X if $\{f(x) : f \in \mathcal{F}\}$ is bounded for each x.

Arzela-Ascoli Theorem, I: Let X be a compact Hausdorff space. If $\mathcal{F}$ is an equicontinuous, pointwise bounded subset of $C(X)$, then it is totally bounded in the uniform metric and is precompact.

Arzela-Ascoli Theorem, II: Let X be a σ-compact, locally compact Hausdorff space. If f_n is an equicontinuous, pointwise bounded sequence in $C(X)$, then there a subsequence that converges to some $f \in C(X)$ uniformly on compact subset.

Cesare Arzelà (1847–1912) or Giulio Ascoli (1843–1896)?
Wikipedia shows this picture for both.

Cesare Arzelà according to St.Andrews

Section 4.5 The Stone-Weierstrass theorem:

Weierstrass's Theorem: The polynomials are uniformly dense in $C([0,1])$.

Throughout this section, X will denote a compact Hausdorff space.

Defn: A subset A of $C(X)$ is said to **separate points** if for every $x, y \in X$ with $x \neq y$ there exists $f \in A$ such that $f(x) \neq f(y)$.

Defn: A is called an algebra if it is a real vector subspace of $C(X)$, such that $fg \in A$ whenever $f, g \in A$.

Defn: If $A \subset C(X, \mathbb{R})$, the A is called a lattice if $f, g \in$ implies $\max(f, g) \in A$ and $\min(f, g) \in A$.

Since the algebra and lattice operations are continuous, one easily sees that if A is an algebra or a lattice, so is its closure A in the uniform metric.

The Stone- Weierstrass Theorem: Let X be a compact Hausdorff space. If A is a closed subalgebra of $C(X, \mathbb{R})$ that separates points, then either $A = C(X, \mathbb{R})$ $A = \{f \in C(X, \mathbb{R} : f(x_0) = O\}$ for some $x_0 \in X$. The first alternative holds iff A contains the constant functions.

Not true for complex functions. The algebra A generated by $\{z^n\}$, $n \geq 0$ on the unit circle separates points, but only contains functions that extend holomorphically into the disk, e.g., $\overline{z} \notin A$. More about this later.

Karl Weierstrass (1815-1897)

Marshall Stone (1903-1989)

Lemma 4.46: Consider $\mathbb{R}^2$ as an algebra under coordinate wise addition and multiplication. Then the only subalgebras are $\mathbb{R}^2$, $\{(O,O)\}$, and the linear spans of $(1,0)$, $(0,1)$, and $(1,1)$.

Proof. These subspaces are evidently subalgebras.

If $A \subset \mathbb{R}^2$ containing a non-zero element (a,b), then $(a^2,b^2) \in A$. If a,b are both non-zero and unequal then these two points are linearly independent and generate the whole plane. Otherwise they generate one of the given subalgebras.

$\square$

Lemma 4.47: For any $\epsilon > 0$ there is a polynomial P on $\mathbb{R}$ such that $P(O) = 0$ and $||x| - P(x)| < \epsilon$ on $[-1, 1]$.

Proof. Proof. Many possible proofs. e.g.,

$$|x| = \sqrt{x^2} \approx \sqrt{x^2 + \epsilon} \approx \sum_0^n a_n(x^2 + \epsilon - 1)^n,$$

where $\sum a_n(y-1)^2$ is the Maclaurin series for $\sqrt{y}$, centered at 1. If $P(0) \neq 0$, replace P by $P - P(0)$. $\square$

Corollary: If $A \subset C(X, \mathbb{R})$ is a closed algebra, then $f \in A \Rightarrow |f| \in A$

Lemma 4.48: If A is a closed subalgebra of $C(X, \mathbb{R})$, and if $f \in A \Rightarrow |f| \in A$, then A is a lattice.

Proof. If $h \in A$ is not the zero function, scale it to have supremum norm 1. By the previous result $|h| \in A$. Thus

$$\max(f, g) = \frac{1}{2}(f + g + |f - g|),$$
$$\min(f, g) = \frac{1}{2}(f + g - |f - g|),$$

are in A. $\square$

Lemma 4.49: Suppose A is a closed sublattice of $C(X, \mathbb{R})$, and if $f \in C(X, \mathbb{R})$. Suppose also that for every $x, y \in X$ there is a $g_{xy} \in A$ that agrees with f at x and y. Then $f \in A$.

Proof. For $\epsilon > 0$, $x, y \in X$ let

$$U_{xy} = \{z \in X : f(z) < g_{xy}(z) + \epsilon\}$$
$$V_{xy} = \{z \in X : f(z) > g_{xy}(z) - \epsilon\}.$$

For a fixed y, U_{xy} is cover of X, so has a finite subcover $\{U_{x_j y}\}_1^n$. Let

$$g_y = \max_j g_{x_j y},$$

and note $f < g_y + \epsilon$ on X. Also $f > g_y - \epsilon$ on

$$V_y = \cap V_{x_j y}.$$

These sets are an open cover of X, so also have a finite subcover $\{V_{k_k}\}$. Let

$$g = \min_h g_{x_k y},$$

and note $g - \epsilon < f < g + \epsilon$ so $|f - g| < \epsilon$. Since A is closed, $f \in A$. $\square$

Proof of Stone-Weierstrass Theorem:

Proof. For $x \neq y \in X$ let

$$A_{xy} = \{(f(x), f(y)) : f \in A\}.$$

This is a subalgebra of $\mathbb{R}^2$ so has one of the forms described earlier.

If $A_{xy} = \mathbb{R}^2$ for all x, y then we are done by the previous lemmas.

A_{xy} can't always be $(0, 0)$ or span of $(1, 1,)$ since it separates points.

Assume A_{xy} is the span of $(1, 0)$ or $(1, 0)$. In either case there is a point where every $f \in A$ vanishes, and only one such point since A separates points. The previous lemma implies $A = \{f \in C(X, \mathbb{R} : f(x_0) = 0\}i$ for some $x_0 \in X$

If A contains all constant functions, there is no such point x_0 so A is all continuous functions on X. $\square$

Cor. 4.50: If B is a subalgebra of $C(X,\mathbb{R})$ that separates points then either B is dense in $C(X,\mathbb{R})$ or dense in

$$\{f \in C(X,\mathbb{R} : f(x_0) = O\}$$

for some $x_0 \in X$.

Complex Stone-Weierstrass Theorem: Suppose X is a compact Hausdorff space. Suppose $A \subset C(X)$ is a closed subalgebra that separates points and that is closed under complex conjugation. Then $A = C(X)$ or $A = \{f \in C(X) : f(x_0) = 0\}$ for some $x_0 \in X$.

Proof. Since $\mathrm{Re}(f) = (f + \overline{f})/2$ and $\mathrm{Im}(f) = (f - \overline{f})/2i$, A contains it real and imaginary parts. Apply SW-theorem to these. and $\square$

Defn: If A is a subalgebra of $C(X)$, a nonempty subset S of X is called an A-antisymmetric set if whenever f is an element of A and the restriction of f to S is real-valued, then the restriction of f to S is constant.

Associated to A is a unique decomposition of X into pairwise disjoint, nonempty, closed, maximal A-antisymmetric sets. Denote this family by $A(X)$.

Errett Bishop's Stone-Weierstrass Theorem 1961: Let X be a compact Hausdorff space, A a uniformly closed subalgebra of $C(X)$ containing the constants. Then $f \in C(X)$ belongs to A iff for for every anti-symmetric set S there is an $F \in A$ so that $f|_S = F|_S$.

Burckel's MMA article on Bishop's theorem

Erret Bishop (1928–1983)

Machado's Theorem 1977: Let X be a compact Hausdorff space, A a uniformly closed subalgebra of $C(X)$ containing the constants. Then

$$\operatorname{dist}(f, A) = \inf_{g \in A} \sup_{x \in X} |f(x) - g(x)| = \sup_{S} \operatorname{dist}(f|_S, A|_S),$$

where supremum on the right is over all anti-symmetric sets S.

There are many generalizations variations. Here is one I proved as a postdoc.

Thm. (C. Bishop 1996) If f is real valued and harmonic on the unit dist, let A be the closed algebra generated by f and the set of bounded holomorphic functions. If $g \in L^\infty(\mathbb{D})$ then

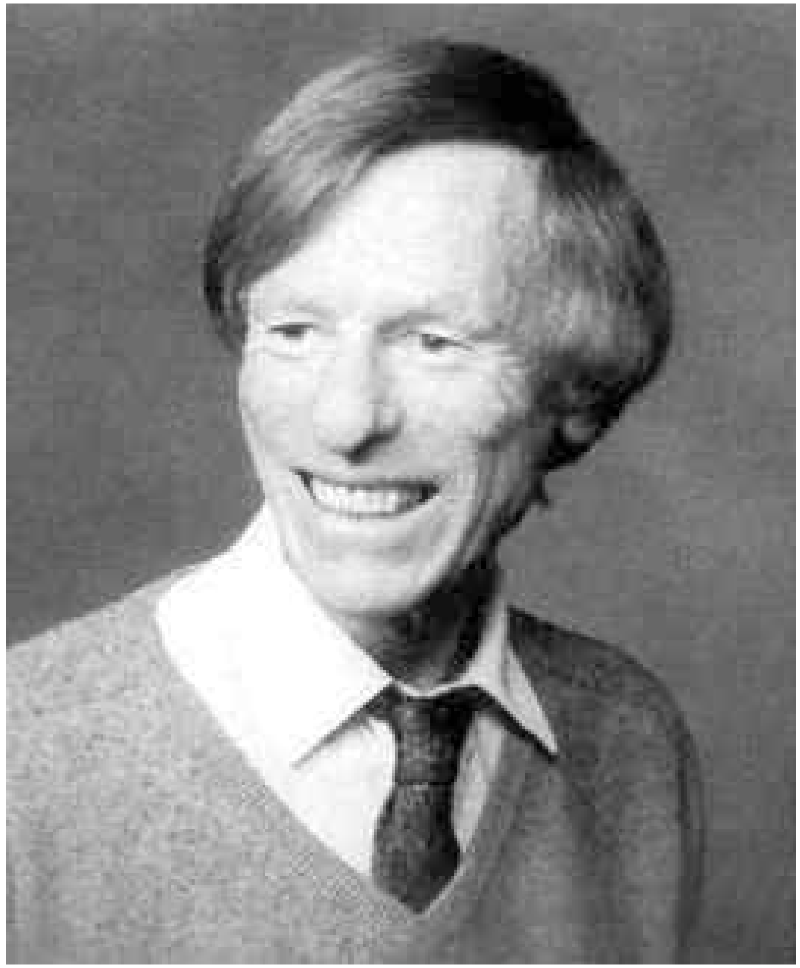
$$\operatorname{dist}(g, A) = \inf_{\delta > 0} \sup_{a \in \mathbb{C}} \operatorname{dist}(g, H^\infty(f^{-1}(D(z, r)))).$$

paper

The closed algebra generated by f and the set of bounded holomorphic functions is usually denoted $H^\infty(\mathbb{D})[f]$. I f is constant on an open subset U of $\mathbb{D}$, the functions in A are clearly holomorphic on U, so the condition is reasonable.

Cor. (Axler-Shields, 1987) If f is a bounded harmonic function on $\mathbb{D}$ that is not holomorphic, then $H^\infty(\mathbb{D})[f]$ contains $C(\overline{\mathbb{D}})$.

Sheldon Axler (1949–present)



Allen Shields (1927-1989)