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REAL ANALYSIS I

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2.5: Product Measures

Main points today:

- Define product measure on product spaces
- Monotone class lemma (useful proof technique)
- Fubini-Tonelli Theorem
- Examples of how thee can fail (if we drop a hypothesis)

Let $(X, \mathcal{M}, \mu)$ and $(X, \mathcal{N}, \nu)$ be measure spaces.

We already discussed product σ-algebras $\mathcal{M} \otimes \mathcal{N}$.

Now we want to define a product measure $\mu \times \nu$.

Defn: a (measurable) **rectangle** is a set of the form $A \times B$ where $A \in \mathcal{M}$ and $B \in \mathcal{N}$

Recall that a elementary family of sets $\mathcal{E}$ (page 23) is a collection satisfying

$\emptyset \in E$

$E, F \in \mathcal{E} \Rightarrow E \cap F \in \mathcal{E}$

if $E \in \mathcal{E}$ then E^c is a finite union of sets in $\mathcal{E}$

Rectangles are an elementary family:

$\emptyset \times \emptyset = \emptyset$

$(A \times B) \cap (E \times F) = (A \cap E) \times (B \cap F)$,

$(A \times B)^c = (X \times B^c) \cup (A^c \times X)$

Recall Proposition 1.7: If $\mathcal{E}$ is an elementary family then the collection of finite unions of members of $\mathcal{E}$ is an algebra (closed under finite unions and complements).

Therefore the collection $\mathcal{A}$ of finite disjoint unions of rectangles is an algebra, and the σ-algebra it generates is $\mathcal{M} \otimes \mathcal{N}$.

Suppose $A \times B$ is a rectangle that is a (finite or countable) disjoint union of rectangles $A_j \times B_j$.

Then for $x \in X$ and $y \in Y$,

$$\chi_A(x)\chi_B(y) = \chi_{A\times B}(x,y) = \sum_j \chi_{A_j \times B_j}(x,y) = \sum_j \chi_{A_j}(x)\chi_{B_j}(y).$$

Integrate over x and use Theorem 2.15 ($\int \sum f_j = \sum \int f_j$ if $f_j \geq 0$),

$$\mu(A)\chi_B(y) = \int \chi_A(x)\chi_B(y)d\mu(x) = \sum_j \int \chi_{A_j}(x)\chi_{B_j}(y)d\mu(x) = \sum_j \mu(A_j)\chi_{B_j}(y)$$

In the same way, integration in y gives

$$\mu(A)\nu(B) = \sum_j \mu(A_j)\nu(B_j).$$

It follows that if $E \in \mathcal{A}$ is the disjoint, finite union of rectangles and we set

$$\pi = \sum_{1}^{n} \mu(A_j)\nu(B_j),$$

then π is well defined and is a pre-measure.

According to Theorem 1.14, π generates an outer measure on $X \times Y$ whose restriction to $\mathcal{M} \otimes \mathcal{N}$ is a measure that extends π.

Easy to check that if μ, ν are σ-finite so is $\mu \times \nu$.

In this case, by Theorem 1.14, $\mu \times \nu$ is the unique measure on $\mathcal{M} \otimes \mathcal{N}$ so that l

$$\mu \times \nu(A \times B) = \mu(A)\nu(B)$$

for all rectangles $A \times B$.

The same construction works for any finite number of factors.

Define a rectangle

$$A_1 \times \cdots \times A_n$$

and set

$$(\mu_1 \times \cdots \times \mu_n)(A_1 \times \cdots \times A_n) = \mu_1(A_1) \cdots \mu_n(A_n).$$

However, for the rest of today we will deal with just two factors.

If $E \subset X \times Y$ we define the x-section and y-sections as

$$E_x = \{y \in Y : (x, y) \in E\}, \qquad E_y = \{x \in X : (x, y) \in E\}.$$

If f is a function on $X \times Y$ we define the x-section and y-section as

$$f_x(y) = f^y(x) = f(x, y).$$

2.34 Proposition:

(a) If $E \in \mathcal{M} \otimes \mathcal{N}$ then $E_x \in \mathcal{N}$ for all $x \in X$ and $E_y \in \mathcal{M}$ for all $y \in Y$.

(b) If f is $\mathcal{M} \otimes \mathcal{N}$ measurable, then f_x is $\mathcal{N}$-measurable for all $x \in X$ and f_y is $\mathcal{M}$-measurable for all $y \in Y$.

Proof: For (a) it suffices to show that the collection of E's for which this is true is a σ-algebra that contains all rectangles, hence contains $\mathcal{M} \otimes \mathcal{N}$.

It clearly contains all rectangles since $(A \times B)_x = B$ or $\emptyset$. Also $(E^c)_x = (E_x)^c$ and $(\cup_j E_j)_x = \cup_j (E_j)_x$, so it is closed under complements and countable unions. Same for y-sections.

For (b) note that $(f_x)^{-1}(B) = (f^{-1}(B))_x$ and $(f^y)^{-1}(A) = (f^{-1}(A))_y$. $\square$

What we just did is a common technique: to show some property holds for all measurable sets, show it holds for a generating sets and show that the property is preserved under complements and countable unions. Thus it holds on some σ-algebra containing the generating sets, and hence on the minimal one.

However, sometime is more convenient to take a "short cut" and only consider nested unions (and intersections) instead of arbitrary ones.

A **monotone class** on a space X is a collection of subsets that is closed under countable increasing unions and countable decreasing intersections, i.e., if $E_1 \subset E_2 \subset \ldots$ are in the collection, so is $\cup_j E_j$ and if $E_1 \supset E_2 \supset \ldots$ are in the collection, so is $\cap_j E_j$.

Any σ-algebra is a monotone class.

Any intersection of monotone classes is a monotone class, so there is a smallest one containing any given collection of subsets.

2.35 The Monotone Class Lemma: If $\mathcal{A}$ is an algebra of subsets of X, then the monotone class $\mathcal{C}$ generated by $\mathcal{A}$ coincides with the σ-algebra $\mathcal{M}$ generated by $\mathcal{A}$.

Proof. Since $\mathcal{M}$ is a monotone class, we have $\mathcal{C} \subset \mathcal{M}$. Thus it is enough to show $\mathcal{C}$ is a σ-algebra (since $\mathcal{M}$ is the smallest containing $\mathcal{A}$).

For $E \in \mathcal{C}$ define

$$C(E) = \{F \in \mathcal{C} : E \setminus F, F \setminus E, E \cap F \in \mathcal{C}\}.$$

Clearly $\emptyset$ and E are in $C(E)$ and and $E \in C(F)$ iff $F \in C(E)$

Also, it is easy to check that $C(E)$ is a monotone class.

If $E \in \mathcal{A}$, then $F \in C(E)$ for all $F \in \mathcal{A}$ because A is an algebra. Thus $A \subset C(E)$, and hence $\mathcal{C} \subset C(E)$.

So if $F \in \mathcal{C}$ then $F \in C(E)$ for every $E \in \mathcal{A}$. But this implies $E \in C(F)$ for all $E \in A$ and hence $\mathcal{A} \subset C(F)$ for any $F \in \mathcal{C}$.

Therefore $\mathcal{C} \subset C(F)$ since these are both monotone classes containing $\mathcal{A}$ and $\mathcal{C}$ is the minimal such, by definition.

Conclusion: If $E, F \in \mathcal{C}$ then so are, $E \setminus F$ and $E \cap F$.

Since $X \in \mathcal{A} \subset \mathcal{C}$, $\mathcal{C}$ is therefore closed under complements. Since

$$(A \cup B) = (A^c \cap B^c)^c$$

$\mathcal{C}$ is also closed under finite unions, hence is an algebra.

If $\{E_j\} \in \mathcal{C}$, then $\cup_1^n E_j \in \mathcal{C}$ for all finite n and since these unions are nested sets, the infinite union is also in $\mathcal{C}$. Thus $\mathcal{C}$ is a σ-algebra. $\square$

2.36 Theorem: Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$, then the functions $x \to \nu(E_x)$ and $y \to \mu(E^y)$ are measurable on X and Y, respectively, and

$$\mu \times \nu(E) = \int \nu(E_x) d\mu(x) = \int \mu(E^y) d\nu(y).$$

Proof. First suppose that μ and ν are finite. Let $\mathcal{C}$ be the set of all $E \in \mathcal{M} \otimes \mathcal{N}$ for which the conclusions of the theorem are true. Since rectangles are an algebra, it will suffice to show

(1) finite unions of rectangles are in $\mathcal{C}$

(2) $\mathcal{C}$ is a monotone class

Then by the monotone class lemma $\mathcal{C}$ equals $\mathcal{M} \otimes \mathcal{N}$.

If $E = A \times B$, is a rectangle, then

$$\nu(E_x) = \chi_A(x)\nu(B), \text{ and } \mu(E^y) = \mu(A)\chi_B(y),$$

so clearly $E \in \mathcal{C}$. Any finite union of rectangles can be written as a finite disjoint union of rectangles.

In this case $x \to \nu(E_x)$ and $y \to \mu(E^y)$ are simple functions, hence measurable, If $E = \cup_j E_j = \cup_j (A_j \times B_j)$ then

$$\begin{aligned} \int \sum \nu(E_x) d\mu(x) &= \int \sum_j \nu(\cup (E_j)_x) d\mu(x) \\ &= \sum_j \int \nu(\cup (E_j)_x) d\mu(x) = \sum_j \int \nu(B_j)\mu(A_j) = \mu(E). \end{aligned}$$

The same for the y-sections. This gives (1).

Next we show $\mathcal{C}$ is a monotone class.

Unions: If $\{E_n\}$ is an increasing sequence in $\mathcal{C}$ and $E = \cup E_n$, then the functions $f_n(y) = \mu((E_n)^y)$ are measurable and increase pointwise to $f(y) = \mu(E^y)$. Hence f is measurable, and by the monotone convergence theorem,

$$\int \mu(E^y)d\nu(y) = \lim \int \mu((E_n)^y)d\nu(y) = \lim \mu \times \nu(E_n) = \mu \times \nu(E).$$

Likewise for $\int \nu(E_x)d\mu(x)\mu\times\nu(E)$. Thus $\mathcal{C}$ is closed under countable increasing unions.

Intersections: Suppose $\{E_n\} \subset \mathcal{C}$ is decreasing and set $E = \cap E_n$. Since

$$\mu(E^y) \leq \mu((E_1)^y) \leq \mu(X) < \infty,$$

the function $y \to \mu((E_n)^y)$ are dominated by the L^1 (constant) function $\mu(X)$. The dominated convergence theorem implies $E \in \mathcal{C}$, so $\mathcal{C}$ is a monotone class. This is (2).

We have now finished the proof for finite measures.

If μ and ν are σ-finite, we can write $X \times Y$ as the union of an increasing sequence $\{X_j \times Y_j\}$ of rectangles of finite measure. If $E \in \mathcal{M} \otimes \mathcal{N}$, the preceding argument applies to $E_j = E \cap (X_j \times Y_j)$. Thus

$$\mu \times \nu(E_j) = \int \chi_{X_j} \nu(E_x \cap Y_j) d\mu(x) = \int \chi_{Y_j} \mu(E^y \cap X_j) d\nu(y).$$

Now apply the Monotone Convergence Theorem to get the result. $\square$

2.37 The Fubini-Tonelli Theorem: Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ σ-finite measure spaces.

(Tonelli) If $f \in L+(X \times Y)$, then the functions $g(x) = \int f_x d\nu$ and $h(y) = \int f^y d\mu$ are in $L^+(X)$ and $L^+(Y)$, respectively, and

$$\int f d(\mu \times \nu) = \int [\int f(x,y) d\nu(y)] d\mu(x) \tag{0.1}$$

$$= \int [\int f(x,y) d\mu(x)] d\nu(ly) \tag{0.2}$$

(Fubini) If $f \in L^1(\mu \times \nu)$, then $f^y \in L^1(\mu)$ for ν a.e. $y \in Y$ and $f_x \in L^1(\nu)$ for μ a.e. $x \in X$. The a.e. defined functions $g(x) = \int f_x d\nu$ and $h(y) = \int f^y d\mu$ are in $L^1(\mu$ and $L^1(\nu)$ respectively, and (0.1) holds.

Proof: Tonelli's theorem reduces to Theorem 2.36 in case f is a characteristic function, and it therefore holds for nonnegative simple functions by linearity.

If $f \in L^+(X \times Y)$, let $\{f_n\}$ be a sequence of simple functions that increase pointwise to f (Theorem 2.10). The corresponding g_n and h_n increase to g and h, so they are measurable.

By the monotone convergence theorem

$$\int g d\mu = \lim \int g_n d\mu = \lim \int f_n d(\mu \times \nu) = \int f d(\mu \times \nu).$$

This proves Tonelli's theorem.

The preceding argument also shows that if $f \in L^+(X \times Y)$ and $\int f d(\mu \times \nu) < \infty$, then $g < \infty$ a.e. and $h < \infty$ a.e. In other words, $f_x \in L^1(\nu)$ a.e. and $f^y \in L^1(\mu)$ a.e.

If $f \in L^1(\mu \times \nu)$ then Fubini's theorem follows by applying these results to the positive and negative parts of the real and imaginary parts of f. $\square$

L^1 is needed in Fubini's theorem. "Checkerboard" example.

σ-finiteness is needed. See Exercise 2.46. Take $X = Y = [0, 1]$, $\mu =$ Lebesgue measure and $\nu =$ counting measure.

E need not be measurable even if every section E_x and E^y is. See Exercise 2.27: well order $[0, 1]$ and let $E = \{(x, y) : x < y\}$.

Product measures of complete measures need not be complete.

For example, take $m \times m$ on $\mathbb{R}^2$. If $E \subset \mathbb{R}$ is non-measurable, then $E \times \{0\} \subset \mathbb{R} \times \{0\}$ is a subset of a measure zero set, but it is not in the product σ-algebra (because there is a section that equals E).

The Tonelli-Fubini theorem can be reformulated to work in the setting of the completion of a product measure. See the textbook. The main difference is that now all the sections are only measurable a.e., not for all x and y.

Lebesgue proved that a function on $\mathbb{R}^d$ is Riemann integrable iff the set of discontinuities has measure zero.

There is a Riemann integrable function on $\mathbb{R}^2$ so that not all of its x an d y sections are Riemann integrable

What if we rotate the axes? Is there a Riemann integrable function on $\mathbb{R}^2$ that is fails to be Riemann integrable on some line in every direction?