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Abstract

The complemented subspace problem asks, in general, which closed subspaces M of a Banach space X are complemented; i.e. there exists a closed subspace N of X such that $X = M \oplus N$? This problem is in the heart of the theory of Banach spaces and plays a key role in the development of the Banach space theory. Our aim is to investigate some new results on complemented subspaces, to present a history of the subject, and to introduce some open problems.

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1 Introduction.

The problem related to complemented subspaces are in the heart of the theory of Banach spaces. These are more than fifty years old and play a key role in the development of the Banach space theory. Our aim is to review of results on complemented subspaces, to present a history of the subject, and to introduce some open problems.

We start with simple observations concerning definition and properties of complemented subspaces. Some useful sources are [8], [17], [28].

Let X be a normed space, M, N be algebraically complemented subspaces of X (i.e. $M + N = X$ and $M \cap N = \{0\}$), $\pi : X \to \frac{X}{M}$ be the quotient map, $\phi : M \times N \to X$ be the natural isomorphism $(x, y) \mapsto x + y$ and $P : X \to M, P(x+y) = x, x \in M, y \in N$ be the projection of X on M along N. Then the following statements are equivalent:

(i) ϕ is a homeomorphism.

(ii) M and N are closed in X and $\pi|_N$ is a homeomorphism.

(iii) M and N are closed and $P : X \to M$ is a bounded projection.

The Subspaces M and N are called topologically complemented or simply complemented if each of the above equivalent statements holds. If N_1, N_2 are complemented subspaces of a closed subspace M, then N_1 and N_2 are isomorphic Banach spaces.

It is known that every finite dimensional subspace is complemented and every algebraic complement of a finite codimension subspace is topologically complemented.

In a Banach space X, applying the closed graph theorem we can establish that two closed subspace are algebraically complemented if and only if they are complemented. Moreover, if M is a closed subspace of X, then M is complemented if and only if the following equivalent assertions hold:

(I) The quotient map $i : M \hookrightarrow X$ has a left inverse as a continuous operator .

(II) The natural projection $\pi : M \to \frac{X}{M}$ has a right inverse as a continuous operator.

l^∞ is complementary in every normed space X containing it isomorphically as a closed subspace [28]. Also, If $c_\circ$ is subspace of a separable Banach space X , then there is a bounded projection P of X onto $c_\circ$ of norm ≤ 2, cf. [44].

Suppose now that F is a retract of a Banach space X, i.e. F is a Banach subspace of X and there is a continuous linear map $\phi : X \to F$ such that for all $x \in F, \phi(x) = x$. Then $C_\circ(X - F) = \{f \in C(X) : f(x) = 0 \text{ for all } x \in F\}$ is complemented in $C(X)$. In fact, by defining $P : C(X) \to C(X)$ by $P(g) = g \circ \phi$, we have $P^2 = P, \|P(g)\| = \underset{x\in X}{Sup} |g(\phi(x))| \leq \|g\|$ and $KerP = \{g \in C(X) | g(\phi(x)) = 0 \text{ for all } x \in X\} = C_\circ(X - F)$.
Hence we may say that "complemented ideal' is the Gelfand dual of "retract closed subspace" (see [31]).

There are non-complemented closed subspaces. For example, let X be the disk algebra, i.e. the space of all analytic functions on $\{z \in \mathbf{C}; |z| < 1\}$ which are continuous on the closure of D. Then the subspace of $C(T)$ consisting of the restictions of functions of X to $T = \{z \in \mathbf{C}; |z| = 1\}$ is not complemented in X (see [18]).

Throughout the paper $c_\circ, c, l_\infty, l_p$ denot the space of all complex sequences $\{x_n\}$ such that $\lim_{n\to\infty} x_n = 0$, $\{x_n\}$ is convergent, $\{x_n\}$ is bounded, and $\sum_{n=1}^{\infty} |x_n|^p < \infty$, respectively. In addition, L_p denotes the L_p-space over the Lebesgue interval $[0, 1]$. The reader is referred to [20] and [26] for undefined terms and notation.

2 Complementary subspace problem and related results.

This problem asks, in general, which closed subspaces of a Banach space are complemented?

In 1937, Murray [32] proved, for the first time, that $l_p, p \neq 2, p > 1$ has non-complemented subspace.

Phillips [38] proved that $c_\circ$ is non-complemented in l^∞. This significant fact has been refined, reproved or generalized by many mathematicians, cf. [37], [16], [42] and [34].

Banach and Mazur showed that all subspaces in $C[0,1]$ which are isometrically isomorphic to l_1 or $L^1[0,1]$ are non-complemented, cf. [43] and [1].

In 1960, Pelczynski [36] showed that complemented subspaces of l_1 are isomorphic to l_1. Köthe [22] generalized this result to the non-separable case.

In 1967, Lindenstrauss [25] proved that every infinite dimensional complemented subspace of l^∞ is isomorphic to l^∞. This also holds if l^∞ is replaced by l_p, $1 \leq p < \infty$, $c_\circ$ or c.

It is shown by Lindenstrauss [24] that if the Banach space X and its closed subspace Y are generated by weakly compact sets (in particular, if X is reflexive), then Y is complemented in X.

In 1971, Lindenstrauss and Tzafriri [26] proved that every infinite dimensional Banach space which is not isomorphic to a Hilbert space contains a closed non-complemented subspace.

Johnson and Lindenstrauss [19] proved the existence of a continuum of non-isomorphic separable $\mathcal{L}^1$-spaces. (An $\mathcal{L}^1$-space is a space X for which X^{**} is a complemented subspace of an L^1-space)

Classically known complemented subspaces of $L_p, 1 < p < \infty$, $p \neq 2$ are

$l_p, l_2, l_p \oplus l_2$ and L_p itself. In 1981, Bourgain, Rosenthal and Schechtman [3] proved that up to isomorphism, there exist uncountably many complemented subspaces of L_p.

It is shown that a complemented subspace M of l_∞^* is isomorphic to l_∞^* provided M is either w^*-closed or isomorphic to a bidual space, cf. [29].

Pisier [39] established that any complemented reflexive subspace of a C^*-algebra is necessarily linearly isomorphic to a Hilbert space.

In 1993, Gowers and Maurey [13] showed that there exists a Banach space X without non-trivial complemented subspaces.

If E is one of the spaces l_p, $(1 \leq p \leq \infty)$ or $c_\circ$, and X is a vector space complemented in E which contains a vector subspace Y complemented in X and isomorphic to E, then X is isomorphic to E. Moreover, each infinite dimensional vector subspace complemented in E is isomorphic to E. Conversely, if Y is a vector subspace of $E = l^2$ or $c_\circ$ which is isomorphic to E, then Y is complemented in E.

If X is an infinite dimensional vector subspace complemented in some space $C(S)$, then X contains a vector subspace isomorphic to $c_\circ$.

Randrianantoanina [40] showed that if X and Y are isometric subspaces of L_p $(p \neq 4, 6, ...)$, and X is complemented in L_p then so is Y. Moreover, the projection constant does not change. This number is defined to be $\inf\{\|T\| : T : L_p \to X$ is a bounded linear projection of L_p onto $X\}$.

The above theorem fails in the case $p \geq 4$ is an even integer, i.e. there exist pairs of isomorphic subspaces X and Y of L_p to itself so that X is complemented and Y is not.

3 Schroeder-Bernstein Problem.

If two spaces are isomorphic to complemented subspaces of each other, are then they isomorphic?

There are negative solutions to this problem.(see [15] and [?])

4 Basis and complemented subspaces.

A Schauder basis for a Banach space X is a sequence $\{x_n\}$ in X with the property that every $x \in X$ has a unique representation of the form $x = \sum_{n=1}^{\infty} \alpha_n x_n; \alpha_n \in \mathbf{C}$ in which the sum is convergent in the norm topology, cf. [20]. For example, the trigonometrical system is a basis in each space $L^p[0,1], 1 < p < \infty$.

Pelczynski [36] showed that any Banach space with a basis is a complemented subspace of an isomorphically unique space.

In 1987, Szarek [45] showed that there is a complemented subspace without basis of a space with a basis and answered therefore to a problem of fifty years old.

5 Approximation property and complemented subspaces.

A Banach space X has the approximation property (AP) if for every $\epsilon > 0$ and each compact subset K of X there is a finite rank operator T in X such that for each $x \in K, \|Tx - x\| < \epsilon$. If there is a constant $C > 0$ such that for each such T, $\|T\| \leq C$, then X is said to have bounded approximation property (BAP), cf. [20]. For example, every Banach space with a basis has BAP.

Pelczynski [36] proved that every Banach space with the BAP can be complementably embedded in a Banach space with a basis.

6 Complemented minimal subspaces.

A Banach space X is called minimal if every infinite dimensional subspace Y of X contains a subspace Z isomorphic to X. For example $c_\circ$ is minimal. If Z is also complemented then X is said to be complementary minimal. Casazza and Odell [5] showed that Tsirelson's space T (see [46] and [12]) have no minimal subspaces.

Casazza, Johnson and Tzafriri [4] showed that the dual T^* of T is minimal but not complementary minimal.

7 quasi-complemented subspaces.

A closed subspace Y of a Banach space X is called quasi-complemented if there exists a closed subspace Z of X such that $Y \cap Z = \{0\}$ and $Y + Z$ is dense in X.

Then such a subspace Z is said to be a quasi-complement of Y. Those notions are first introduced by Murray [33].

Every closed subspace of l_∞ is quasi-complemented, cf. [42]. Also Mackey [27] proved that in a separable Banach space every subspace is quasi-complemented.

Rosenthal [41] showed that if X is a Banach space, Y is a closed subspace of X, Y^* is W^*-separable and the annihilator $Y^\perp$ of Y in X^* has an infinite dimensional reflexive subspace, then Y is quasi-complement in X.

8 Weakly complemented subspaces.

A closed subspace of a Banach space X is called weakly complemented if the dual i^* of the natural embedding $i : M \hookrightarrow X$ has a right inverse as a bounded operator.

For example, $c_\circ$ is weakly complemented in l_∞, not complemented in l_∞ (see [47]).

If M is complemented in X with the corresponding projection P, then the adjoint of $id_X - P$ is a projection in $B(X)$ with the range $M^o = \{f \in X^*; f|_M = 0\}$. Hence M is weakly complemented in X.

9 contractively complemented subspaces.

As mentioned before, a closed subspace Y of a Banach space X is said to be complemented if it is the range of a bounded linear projection $P : X \to X$. If $\|P\| = 1$, Y is called a contractively complemented or 1-complemented subspace of X.

Let X be a Banach space with $\dim X \geq 3$. Then X is isometrically isomorphic to a Hilbert space iff every subspace of X is the range of a projection of norm 1 (see [21] and [2]).

In 1969, Zippin [48] proved that every saparable infinite dimensional L_1-predual space (i.e a Banach space whose dual is isometric to $L_1(\mu)$ for some measure space (Ω, Σ, μ))) contains a contractively complemented subspace isomorphic to $c_\circ$.

Lindenstrauss and Lazar [23] proved that X contains a contractively complemented subspace isometric to some space $C(S)$ when X^* is non-separable.

Question. Let X be a Banach space and $T : X \to X$ be an isometry. Is the range of T is contractively complemented in X?
In Hilbert and $L^p, (1 \leq p < \infty)$ spaces, we have an affirmative answer. In case $C[0,1]$, however, it may happen that the range of an isometry is not complemented, cf. [9].

Pisier [39] proved that if M is a Von Neumann subalgebra of $B(H)$ which is complemented in $B(H)$ and isomorphic to $M \otimes M$, then M is contractively complemented.

10 Prime Banach spaces and complemented subspaces.

A Banach space X is called prime if each infinite dimensional complemented subspace of X is isomorphic to X, cf. [26].

Pelczynski [36] proved that $c_\circ$ and l_p $(1 \leq p < \infty)$ are prime. Lindenstrauss [25] proved that l^∞ is also prime. Gowers and Maurey [13] constructed some new prime spaces.

11 Complemented subspaces of topological products and sums of Banach spaces.

Metafune and Moscatelli [30] proved that when X is one of the Banach spaces $l_p(1 \leq p \leq \infty)$ or $c_\circ$, then each infinite dimensional complemented subspace of X^N is isomorphic to one of the spaces $\omega, \omega \times X^N$ or X^N, where $\omega = K^N$ (K is the scaler field) and X^N is the product of countably many copies of X.

In [11], the authors obtained a complete description of the complemented subspace of the topological product l_∞^m where m is an arbitrary cardinal number.

Every complemented subspace of a product $V = \prod_{i \in I} X_i$ of Hilbert spaces is isomorphic to a product of Hilbert spaces (I is a set of arbitrary cardinal), cf. [10].

Ostraskii [35] showed that not all complemented subspaces of countable topological products of Banach spaces are isomorphic to topological products of Banach spaces.

Chigogidze [6] proved that complemented subspaces of a locally convex direct sum of arbitrary collection of Banach spaces are isomorphic to locally convex direct sum of complemented subspaces of countable subsums.

Chigogidze [7] proved that a complemented subspace of an uncountable topological product of Banach spaces is isomorphic to a topological product of complemented subspaces of countable subproducts and hence isomorphic to a topological product of Frechet spaces.

12 Some interesting problems.

The following problems in this area arise:

1) Given a Banach space X, characterize the isomorphic types of its complemented subspaces.

2) Given a Banach space X, characterize the isomorphic types of such Banach space Z that every vector subspace of Z isomorphic to X is complemented in Z.

3) Is every complemented vector subspace of $C(S)$ isomorphic to some $C(S_1)$?

4) If a Banach space X is complemented in every Banach space containing it, is X isomorphism to some $C(S)$ over a Stone space S? (A space is Stonian if the closure of every open set is open)

5) Does every complemented subspace of a space with an unconditional basis have an unconditional basis? Recall that an unconditional basis for a Banach space is a basis $\{x_n\}$ such that every permutation of $\{x_n\}$ is also a basis or equivalently, the convergence of $\sum \alpha_n x_n$ implies the convergence of every rearrangement of the series, cf. [20].

6) If a von Neumann algebra is a complemented subspace of $B(H)$, is it then injective?

7) Are $l_p, 1 \leq p \leq \infty$ and $c_\circ$ the only prime Banach spaces with an unconditional basis? is still open.

Remark. Some pieces of information are taken from Internet-based resources without mentioning the URL's.

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