Exercise 1.11: Consider $[0,1]$ with the metric $d(x,y) = \sqrt{|x-y|}$. Show that the interval has dimension 2 with this metric.

Solution 1.11 The problem does not specify Hausdorff or Minkowski dimension, so we will consider both.

First, we give an upper bound for Minkowski dimension. Cover $I = [0,1]$ by n intervals of the form $[\frac{k}{n}, \frac{k+1}{,} n]$ for $k = 0, \dots, n-1$. Each of this has diameter $1/\sqrt{n}$ in the given metric, so $N(I, 1/\sqrt{n}) = n$. Hence $N(I,\epsilon) = n$ if $n^{-1/2} \leq \epsilon \leq (n-1)^{1/2}$. Hence

$$\frac{1}{2}\log(n-1) \leq \log 1/\epsilon \leq \frac{1}{2}\log n,$$

and so

$$\limsup_{\epsilon\to\infty} \frac{\log N(I,\epsilon)}{\log 1/\epsilon} \leq \limsup_{n\to\infty} \frac{\log n}{\frac{1}{2}\log n - 1} = 2.$$

Therefore the upper Minkowski dimension is at most 2.

Next, we give a lower bound for the Hausdorff dimension. Consider usual Lebesgue measure m on $I = [0,1]$. A subinterval $J = [x,y]$ gets measure $|x-y|$ but has diameter $\sqrt{|x-y|}$ (remember the unusual metric). Thus for every interval J,

$$m(J) = \mathrm{diam}(J)^2.$$

By the mass distribution principle, the Hausdorff dimension is at least 2.

Since the lower bound for Hausdorff dimension and the upper bound for the upper Minkowski dimension agree, and the lower Minkowski dimension must be trapped between them, all three numbers agree. Thus Minkowski dimension exists and equals 2, which is the same as the Hausdorff dimension.