HISTORY OF THE RIEMANN MAPPING THEOREM

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The Riemann mapping theorem, that *an arbitrary simply connected region of the plane can be mapped one-to-one and conformally onto a circle*, first appeared in the Inaugural dissertation of Riemann (1826–1866) in 1851. The theorem is important, for by it a result proved for the circle can often be transformed from the circle to a more general region. The proof is difficult, as involving both behavior of a function in the small (conformal mapping) and behavior in the large (one-to-one mapping). Riemann's proof was open to criticism and in the following decades numerous mathematicians sought for a proof, e.g., H. A. Schwarz (1843–1921), A. Harnack (1851–1888), H. Poincaré (1854–1912), etc., until the first rigorous proof was given in 1900 by W. F. Osgood. The proof of Osgood represented, in my opinion, the "coming of age" of mathematics in America. Until then, numerous American mathematicians had gone to Europe for their doctorates, or for other advanced study, as indeed did Osgood. But the mathematical productivity in this country in quality lagged behind that of Europe, and no American before 1900 had reached the heights that Osgood then reached.

William Fogg Osgood (1864–1943) was born in Boston in 1864, graduated from Harvard College in 1886, stayed in Cambridge for a year of graduate work, and then went to Göttingen with a Harvard fellowship for further study, especially with Felix Klein (1849–1925). According to gossip, Osgood became so enamored of a Göttingen lady that his work suffered and Klein sent him to Erlangen for his doctorate. In any case, he was accorded the degree from Erlangen in 1890 for a thesis on Abelian integrals, and one or two days later he married the girl in Göttingen, and one or two days still later they sailed for the United States of America. His

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Professor Walsh received his Harvard Ph. D. under Maxime Bôcher and George David Birkhoff. He continued at Harvard as Instructor through Perkins, Professor of Mathematics and became Professor Emeritus in 1966; since then he has been at the Univ. of Maryland. He has spent leaves of absence at the Sorbonne, the Univ. of Munich, the Institute for Advanced Study, and has spent several sabbatical leaves in Paris and Jerusalem.

He is a Fellow of the American Academy of Arts and Sciences and a Member of the National Academy of Sciences. Both the SIAM Journal on Numerical Analysis and the Journal of Approximation Theory have dedicated volumes to Joseph Walsh. His main research is on zeros, extremal problems, and approximations by polynomials and orthogonal functions. He is widely known for his invention of the Walsh functions.

His publications include *Interpolation and Approximation* (Amer. Math. Soc. Coll. Series, 1935, Russian Tranlation — 1961), *Location of Critical Points of Analytic and Harmonic Functions* (Amer. Math. Soc. Coll. Series, 1950), *Approximation by Polynomials* (Paris, 1935), *Approximation by Bounded Analytic Functions* (Paris 1960). *The Theory of Splines and their Application* (with J. H. Ahlberg and E. N. Nilson, Academic Press, New York, 1967), *A Bibliography on Orthogonal Polynomials* (with J. S. Shohat and Einar Hille, National Research Council, Bulletin, Washington, D. C. 1940), and *A Rigorous Treatment of Maximum-Minimum Problems in the Calculus* (Heath, 1962). *Editor.*

early mathematical work was also of high quality. During the 1890's he was Lebesgue's forerunner in the study of sequences of functions of a real variable. Osgood taught at Harvard from 1890 until his retirement in 1933.

Osgood seems not to have received the recognition for his work that he deserves. For instance, C. Carathéodory and G. Julia each wrote a book on conformal mapping without mention of the name of Osgood.

We proceed now with the proof of Riemann's theorem!

By a **simply connected region** Riemann understood a region bounded by a simple closed curve, and before him special mappings by simple functions were well known. We assume the given region to be bounded, which may require an elementary preliminary transformation. Let us examine Riemann's proof (based on Dirichlet's Principle) and postpone discussion of its validity.

Mapping of a region T onto a circle is equivalent to the existence of **Green's function for** T, namely a function $G(z)$ such that

(1) $G(z)$ is harmonic in T except at the origin 0, assumed interior to T;

(2) in the neighborhood of 0 the function takes the form $G(z) \equiv G_1(z) + \log r$, where $r = |z|$ and $G_1(z)$ is harmonic throughout T;

(3) $G(z)$ is continuous and equal to zero at every point of the boundary C of T.

These three conditions determine $G(z)$ uniquely. Green's function for a region T is invariant under one-to-one conformal mapping of T.

If the function $w = \phi(z)$ maps T (Figure 1) onto $|w| < 1$ so that $\phi(0) = 0$, then we clearly have

$$\phi(z) = e^{G(z)+iH(z)}$$

where $H(z)$ is conjugate to $G(z)$ in T, for each of the conditions (1), (2), (3), is satisfied by $G(z)$ as thus defined. Conversely, if $G(z)$ is Green's function for T with pole in 0, then every point of T is transformed by $w = \phi(z)$ into a point $|w| < 1$. Each locus L_r: $|\phi(z)| = r$, $0 < r < 1$ in T bounds two subregions of T, where $G(z) > \log r$ and $G(z) < \log r$ respectively; the locus L_r has no multiple points and

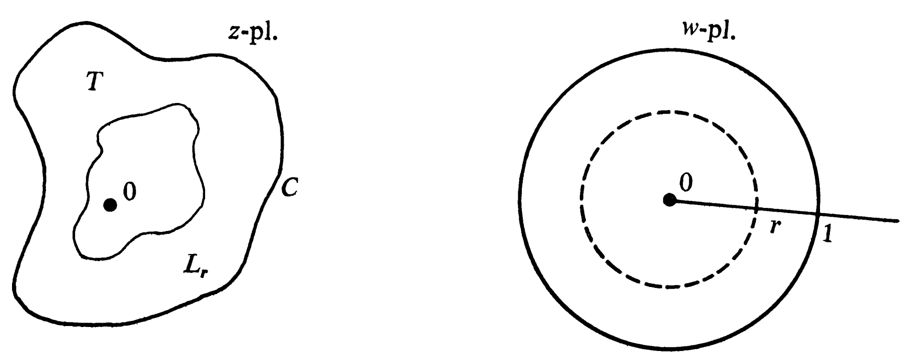

FIG. 1

separates 0 and C. On L_r we have $\partial G/\partial n \neq 0$, where n is the interior normal for the latter subregion, whence

$$\int_{L_r} \frac{\partial G}{\partial n} ds = \arg w \,|_{L_r} = \int_{L_r} \frac{\partial H}{\partial s} ds = \int_{L_r} \frac{\partial \log r}{\partial n} ds = 2\pi,$$

so the transformation $w = \phi(z)$ defines a one-to-one map of T onto $|w| < 1$.

If T is given, the determination of $G(z)$ requires the solution of the Dirichlet problem for T with the prescribed boundary values $\log r$ on C, a problem that Riemann treated by means of Dirichlet's principle. The physical evidence for the existence of $G(z)$ is great, for in the steady two-dimensional flow of heat, the temperature is a harmonic function provided T is a uniform body whose continuous boundary temperatures on C are prescribed.

The Dirichlet integral defined for a function $u(x, y)$ given in a region T is defined as

$$D(u) = \iint_T \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\,dy \quad (\geqq 0).$$

We compare this integral with the corresponding integral where $u(x, y)$ is replaced by $u(x, y) + \varepsilon \cdot v(x, y)$, where $v(x, y)$ vanishes on the boundary C of T. Thus we have, *to study the function* $u(x, y)$ *with given boundary values minimizing* $D(u)$,

$$D(u + \varepsilon v) = \iint_T \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\,dy + 2\varepsilon \iint_T \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) dx\,dy$$
$$+ \varepsilon^2 \iint_T \left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] dx\,dy.$$

Considered as a function of ε, this second term on the right must be zero, namely,

$$\iint_T \frac{\partial}{\partial x}\left[\left(v\,\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(v\,\frac{\partial u}{\partial y}\right)\right] dx\,dy - \iint_T v\nabla^2 u\,dx\,dy = 0$$

for all choices of the arbitrary function v. The former of these two integrals reduces to two contour integrals over C with v ($= 0$ on C) as a factor of the integrand. Thus for the function u minimizing $D(u)$, $\nabla^2 u = 0$ throughout T, and u is harmonic in T. "The function solving the boundary value problem is the function minimizing $D(u)$."

This "proof," although accepted by Riemann, is obviously open to various objections:

(1) The treatment has a meaning only if C has certain properties of smoothness and differentiability.

(2) The fact that $D(u)$ has a non-negative greatest lower bound does not show the existence of a *minimum* (Weierstrass).

(3) The fact that $D(u) < \infty$ for some $u(x, y)$ satisfying the given boundary values needs to be shown (Prym 1871, Hadamard 1906).

It is convenient to assume that T is bounded; if not, we may use the transformation $w = \sqrt{(z-\alpha)/(z-\beta)}$, where α and β are two distinct boundary points of T. Then T in the z-plane corresponds to two regions T_1 and T_2 on the w-sphere, one-to-one conformal images of T, which have no common point. If two such regions do not exist, a point w_1 in T_1 can be joined to a point w_1 by a path in T_1 separating $w = 0$ and $w = \infty$, so there is a closed curve in T separating α and β, and T is not simply connected. Inversion of a point of T_2 to infinity now maps T_1 onto a bounded region.

We mention here several results that we shall need for discussion of Osgood's proof.

(1) **Axel Harnack's Theorem (1887).** If a function u_n is harmonic in a region T for all sufficiently large values of n, and if u_n increases at all points of T when n increases; if furthermore at a single point of T u_n approaches a (finite) limit when n becomes infinite; then u_n converges at all points of T, to a function harmonic throughout T. (It is reported that when Harnack first told Felix Klein of this theorem, the latter refused to believe its validity.)

(2) **H. A. Schwarz.** Green's function exists for a simply connected region T bounded by a finite number of analytic arcs. (Schwarz used the alternating method, due to C. Neumann.)

(3) **Lemma.** If the bounded region T contains the closure of the region T_1, and if O lies in T_1, then the respective Green's functions g and g_1 with poles in O for T and T_1 satisfy the inequality $g > g_1 > 0$ in T_1. For the difference $g - g_1$ is harmonic in T_1, and $g_1 = 0$, $g > g_1$, on the boundary of T_1, whence $g - g_1 > 0$, $g - g_1 \not\equiv 0$, throughout T_1.

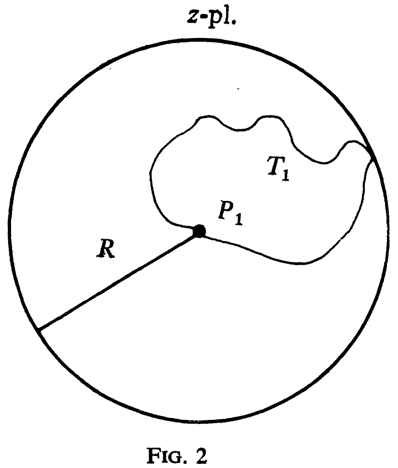

FIG. 2

(4) Given a region T_1, it can be exhausted by a monotonic sequence of subregions, composed for instance of adjacent squares whose sides are parallel to the coordinate axes.

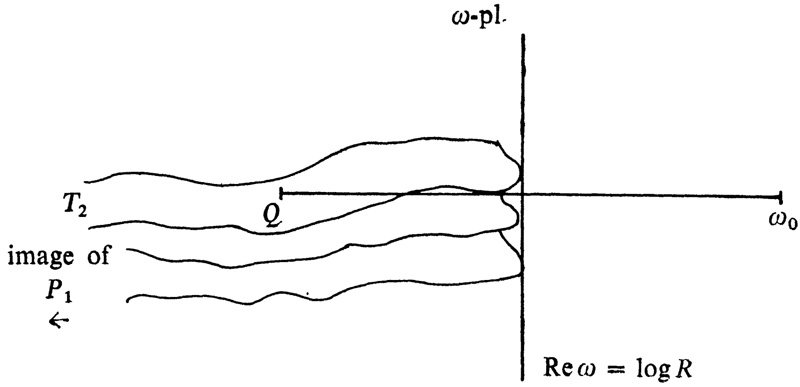

FIG. 3

Given, now, (Figure 2) a bounded simply connected region T_1 of the z-plane, we show that T_1 can be transformed into a region T of the w-plane in such a manner that a given boundary point P_1 of T_1 corresponds to a point P of a circle Γ which contains T. Let T_1 be considered to lie on the Riemann surface for $\omega = \log z$ with P_1 at $z = 0$. The image T_2 in the ω-plane of T_1 consists (Figure 3) of an infinite number of images of T_1, each the translation of another such region by the vector

z	ω	w
P_1	∞	P
T_1	T_2	T
	ω_0	∞
	Q	O_w

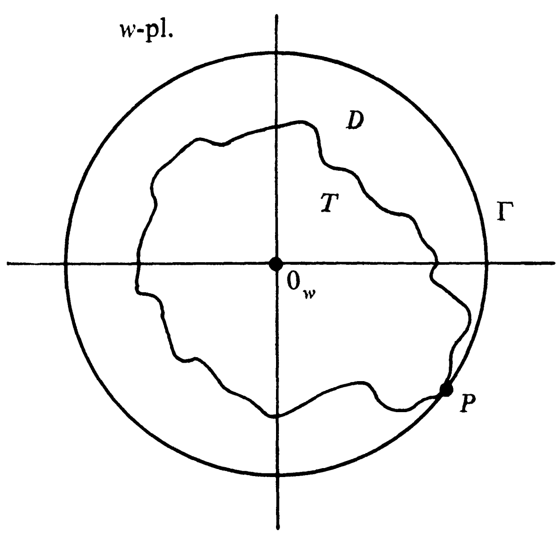

FIG. 4

$\omega = \pm 2\pi i$. In each such region the point at infinity $\omega = \infty$ corresponds to P_1, for all boundary points of T_1 in the region $|z| < \varepsilon$ correspond to points ω with $\operatorname{Re}\omega < \log \varepsilon$. Let $|z| < R$ be the smallest circular disk with center P_1 containing T_1; then all points of T_2 lie in the half-plane $\operatorname{Re}\omega < \log R$. A linear transformation carrying to infinity in the w-plane a finite point ω_0, $\operatorname{Re}\,\omega_0 > \log R$; carries T_2 into a region T of the w-plane (Figure 4) which lies in a circular disk D (image of $\operatorname{Re}\omega < \log R$) whose boundary passes through the image P of P_1.

It may be noted too that an arbitrary point Q of T_2 with $\operatorname{Im} Q = \operatorname{Im}\,\omega_0$ can be chosen so that Q is simultaneously carried into O_w in the w-plane. The point O_w in T is then the center of D.

Let D_n be a monotonic sequence of subregions of T containing O_w, and each with a Green's function g_n with pole in O_w, with $D_{n+1} \supset D_n$, exhausting T. Let g_0 be Green's function for D with pole in O_w; then $g_n < g_0$ in D_n. Let $g = \lim_{n\to\infty} g_n$, defined (Harnack) and harmonic throughout T except in 0. Then g is Green's function for T; we have $0 < g_n < g_0$, $0 < g \leqq g_0$. Suppose $P_k \in T$, $P_k \to P$. Since

$$\lim_{P_k \to P} g_0(P_k) = 0 \text{ for } P_k \in D,$$

we also have

$$\lim_{P_k \to P} g(P_k) = 0, \ g(P) = 0,$$

and this shows the existence of Green's function for T and thus completes Osgood's proof of Riemann's theorem.

We have not mentioned the work of Hilbert (1862–1943), who gave a treatment of Riemann's theorem in weakened form by new methods of the Calculus of Variations, commencing about 1900. This general problem in the Calculus of Variations was presented as Problem 20 in his famous Paris lecture of 1900. He suggested in particular thesis topics on the subject for several American doctoral students in Göttingen: C. A. Noble, E. R. Hedrick, and Max Mason. However, Hilbert's method required certain smoothness properties of the boundary and of the limit function, and was thus less general than the idea of the original Dirichlet principle and less general than Osgood's proof. A new method of proof, based on function theoretic rather than potential theoretic properties, was developed by F. Riesz and L. Fejér, published in 1923 by T. Radò. Montel's theory of normal families was used, and a lemma due to Koebe. This is the standard modern proof.

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THE FIRST U. S. A. MATHEMATICAL OLYMPIAD

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At its meeting on September 1, 1971, the Mathematical Association of America agreed to sponsor a U. S. A. Mathematical Olympiad in addition to the Annual High School Mathematics Examination. The purpose of the Olympiad was to attempt to discover secondary school students with superior mathematical talent — who possessed mathematical creativity and inventiveness as well as competence in computational techniques. Participation was to be limited to about 100 students selected from the Honor Roll on the High School Mathematics Examination, plus a few students of superior ability selected from those states which did not participate in the High School Mathematics Examination. The Olympiad itself was to consist of five essay-type problems requiring mathematical power on the part of the participants. The committee responsible for conducting the Olympiad consisted of Samuel L. Greitzer, Rutgers University, Alfred Kalfus, Babylon High School, Murray S. Klamkin, Ford Motor Company, and Nura D. Turner, SUNY at Albany.

Invitations were sent to 106 students on April 14, 1972, and 100 students took the Olympiad on May 9, 1972. The committee which prepared the Olympiad consisted of Murray Klamkin, D. J. Newman, Yeshiva University and Abraham Schwartz, CUNY. The Olympiad is reproduced below. (Solutions have been provided at the end of this article.)

THE FIRST U. S. A. MATHEMATICAL OLYMPIAD

MAY 9, 1972

1. The symbols $(a, b, \ldots, g)$ and $[a, b, \ldots, g]$ denote the greatest common divisor and the least common multiple, respectively, of the positive integers $a, b, \ldots g$. For