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Introduction to quasiconformal mappings in n-space

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Abstract. We give an introduction to quasiconformal mappings in the Euclidean space $\mathbb{R}^n$.

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Contents

1. Introduction: Mercator's map	240
2. History and background	241
3. Preliminaries	242
ACL^p functions	242
Conformal mappings	242
Möbius transformations	242
4. Modulus of a path family	244
Ring domains	248
Modulus in conformal mappings	250
Capacity of a condenser	251
Sets of zero capacity	252
Spherical symmetrizations	252
Canonical ring domains	253
Spherical metric	256
5. Quasiconformal mappings	256
Examples	257
6. An application of the modulus technique	258
References	259

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1. Introduction: Mercator's map

Perhaps the greatest cartographer of the time, Gerardus Mercator (5 March 1512 – 2 Dec 1594) was born Gerhard Kremer of German parents in the town of Rupelmonde near Antwerp. Like many other intellectuals of his time, he Latinized his German name, which meant "merchant", and changed it to the name Mercator which means "world trader". Mercator was a mapmaker, scholar, and religious thinker. His interests ranged from mathematics to calligraphy and the origin of the universe. Mercator studied mathematics in Louvain under the supervision of mathematician and astronomer Gemma Frisius.

FIGURE 1: Gerardus Mercator (source: Wikipedia) and a World map using the Mercator projection.

The Mercator map is defined by the formula

$$(x, y) = \big(\lambda, \log\big(\tan(\pi/4 + \phi/2)\big)\big),$$

where ϕ is the *latitude* and λ is the *longitude* of the point on the sphere. Mercator published the first map using this projection in 1569, a wall map of the world on 18 separate sheets entitled: "New and more complete representation of the terrestrial globe properly adapted for its use in navigation." The projection did not become popular until 30 years later (1599), when Edward Wright published an explanation of it. An important property of the Mercator projection is that it is conformal, i.e. the angles are preserved.

FIGURE 2: India and Finland in the Mercator projection.

The Mercator projection is not without flaws, however. For example, from the picture above one might conclude that India is approximately twice as large as Finland. Actually, India's land area is $3,287,590$ km^2, almost ten times that of Finland ($338,145$ km^2). This example also illustrates the reasons why we are mainly interested in the local distortion of the geometry in this theory.

2. History and background

Conformal mappings play extremely important role in complex analysis, as well as in many areas of physics and engineering. The class of conformal mappings turned out to be too restrictive for some problems. Quasiconformal mappings were introduced by H. Grötzsch provide more flexibility in 1928. Important results were also obtained by O. Teichmüller and L. V. Ahlfors [1]. A comprehensive survey on quasiconformal mappings of the complex plane is [16]. See also [15].

By the Riemann mapping theorem a simply-connected plane domain with more than one boundary point can be mapped conformally onto the unit disk $\mathbf{B}^2$. On the other hand, Liouville's theorem says that the only conformal mappings in $\mathbb{R}^n$, $n \geq 3$, are the Möbius transformations. Hence the plane theory of conformal mappings does not directly generalize to the higher dimensions.

Quasiconformal maps were first introduced in higher dimensions by M. A. Lavrent'ev in 1938. The systematic study of quasiconformal maps in $\mathbb{R}^n$ was begun by F. W. Gehring [5] and J. Väisälä [20] in 1961. Since then the theory and it's generalizations have been actively studied [3, 4, 21, 23]. Generalizations include quasiregular [18, 22, 19] and quasisymmetric mappings, and recently the mappings of finite distortion [13] and the quasiconformal mappings in the metric spaces [10, 11, 12].

Quasiconformal mappings in $\mathbb{R}^n$ are natural generalization of conformal functions of one complex variable. Quasiconformal mappings are characterized by the property that there exists a constant $C \geq 1$ such that the infinitesimally small spheres are mapped onto infinitesimally small ellipsoids with the ratio of the larger "semiaxis" to the smaller one bounded from above by C.

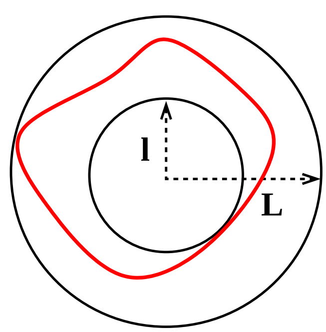

FIGURE 3: Image of a small sphere.

For a comprehensive historical review of the theory of quasiconformal mappings in both plane and space settings, see [2]. A survey of the theory of quasiconformal mappings is given in [8] (see also [14]). This presentation is for the most parts based on [7], [21] and [22].

3. Preliminaries

We shall follow standard notation and terminology adopted from [21], [22] and [19]. For $x \in \mathbb{R}^n$, $n \geq 2$, and $r > 0$ let $\mathbf{B}^n(x,r) = \{z \in \mathbb{R}^n : |z-x| < r\}$, $S^{n-1}(x,r) = \partial \mathbf{B}^n(x,r)$, $\mathbf{B}^n(r) = \mathbf{B}^n(0,r)$, $S^{n-1}(r) = \partial \mathbf{B}^n(r)$, $\mathbf{B}^n = \mathbf{B}^n(1)$, $\mathbf{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$, $\mathbf{B}^n_+ = \mathbf{B}^n \cap \mathbf{H}^n$, and $S^{n-1} = \partial \mathbf{B}^n$. For $t \in \mathbb{R}$ and $a \in \mathbb{R}^n \setminus \{0\}$, $P(a,t) = \{x \in \mathbb{R}^n : x \cdot a = t\} \cup \{\infty\}$, is a hyperplane in $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ perpendicular to the vector a and at distance $t/|a|$ from the origin. The surface area of S^{n-1} is denoted by ω_{n-1} and Ω_n is the volume of $\mathbf{B}^n$. It is well known that $\omega_{n-1} = n\Omega_n$ and that

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$$

for $n = 2, 3, \ldots$, where Γ is Euler's gamma function. The standard coordinate unit vectors are denoted by $e_1, \ldots, e_n$. The k-dimensional Lebesgue measure is denoted by m_k. For $k = n$ we omit the subscript and denote the Lebesgue measure on $\mathbb{R}^n$ simply by m.

For nonempty subsets A and B of $\overline{\mathbb{R}}^n$, we let $d(A) = \sup\{|x-y| : x, y \in A\}$ be the diameter of A, $d(A,B) = \inf\{|x-y| : x \in A, y \in B\}$ the distance between the sets A and B, and in particular $d(x,B) = d(\{x\}, B)$.

ACL^p **functions.** Let Q be a closed n-interval $\{x \in \mathbb{R}^n : a_i \leq x_i \leq b_i, i = 1, \ldots, n\}$. A function $f\colon Q \to \mathbb{R}^m$ is called ACL (*absolutely continuous on lines*) if f is continuous and if f is absolutely continuous on almost every line segment in Q parallel to one of the coordinate axes. Let U be an open set in $\mathbb{R}^n$. A function $f\colon U \to \mathbb{R}^m$ is ACL if $f|Q$ is ACL for every closed n-interval $Q \subset U$. Such a function has partial derivatives $D_i f(x)$ a.e. in U, and they are Borel functions [21, 26.4]. If $p \geq 1$ and the partial derivatives of f are locally L^p-integrable, f is said to be in ACL^p or in ACL$^p(U)$.

Conformal mappings. Let G, G' be domains in $\mathbb{R}^n$. A homeomorphism $f\colon G \to G'$ is called *conformal* if f is in $C^1(G)$, $J_f(x) \neq 0$ for all $x \in G$, and $|f'(x)h| = |f'(x)||h|$ for all $x \in G$ and $h \in \mathbb{R}^n$. If G, G' are domains in $\overline{\mathbb{R}}^n$, a homeomorphism $f\colon G \to G'$ is conformal if its restriction to $G \setminus \{\infty, f^{-1}(\infty)\}$ is conformal.

Möbius transformations. A *Möbius transformation* is a mapping $f\colon \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ that is composed of a finite number of the following elementary transformations:

(1) Translation: $f_1(x) = x + a$.
(2) Stretching: $f_2(x) = rx$, $r > 0$.
(3) Rotation: f_3 is linear and $|f_3(x)| = |x|$ for all $x \in \mathbb{R}^n$.

(4) Reflection in plane $P(a,t)$:

$$f_4(x) = x - 2(x \cdot a - t)\frac{a}{|a|^2}, \qquad f_4(\infty) = \infty.$$

(5) Inversion in a sphere $S^{n-1}(a,r)$:

$$f_5(x) = a + \frac{r^2(x-a)}{|x-a|^2}, \qquad f_5(a) = \infty, \qquad f_5(\infty) = a.$$

In fact every Möbius transformation can be expressed as a composition of a finite number of reflections and inversions. It is easy to see that every elementary transformation, and hence every Möbius transformation, is conformal.

Let a,b,c,d be distinct points in $\mathbb{R}^n$. We define the *absolute (cross) ratio* by

$$|a,b,c,d| = \frac{|a-c|\,|b-d|}{|a-b|\,|c-d|}. \tag{3.1}$$

This definition can be extended for $a,b,c,d \in \overline{\mathbb{R}}^n$ by taking limit.

An important property of Möbius transformations is that they preserve the absolute ratios, i.e.

$$|f(a), f(b), f(c), f(d)| = |a,b,c,d|,$$

if $f\colon \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ is a Möbius transformation. In fact, a mapping $f\colon \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ is a Möbius transformation if and only if f preserves all absolute ratios.

Let $a^* = a/|a|^2$ for $a \in \mathbb{R}^n \setminus \{0\}$, $0^* = \infty$ and $\infty^* = 0$. Fix $a \in \mathbf{B}^n \setminus \{0\}$. Let

$$\sigma_a(x) = a^* + r^2(x-a^*)^*, \qquad r^2 = |a|^2 - 1$$

be an inversion in the sphere $S^{n-1}(a^*,r)$ orthogonal to S^{n-1}. Then $\sigma_a(a) = 0$, $\sigma_a(a^*) = \infty$. Let p_a denote the reflection in the $(n-1)$-dimensional plane $P(a,0)$ through the origin and orthogonal to a, and define a sense preserving Möbius transformation by $T_a = p_a \circ \sigma_a$. Then $T_a(\mathbf{B}^n) = \mathbf{B}^n$ and $T_a(a) = 0$. For $a = 0$ we set $T_a = id$, i.e. the identity map.

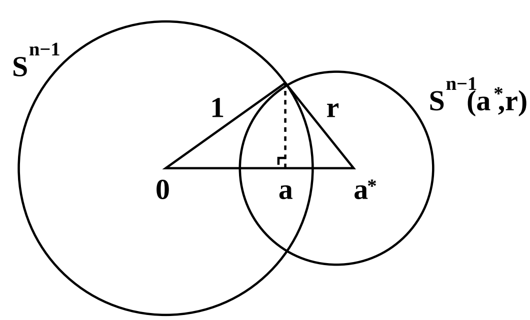

FIGURE 4: Construction of the Möbius transformation T_a.

4. Modulus of a path family

A path in $\mathbb{R}^n$ is a continuous mapping $\gamma\colon \Delta \to \mathbb{R}^n$, where Δ is a (possibly unbounded) interval in $\mathbb{R}$. The path γ is called closed or open according as Δ is compact or open. The locus $|\gamma|$ of γ is the image set $\gamma\Delta$.

Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a closed path. The length $\ell(\gamma)$ of the path γ is defined by means of polygonal approximation (see [21], pages 1-8). The path γ is called rectifiable if $\ell(\gamma) < \infty$ and locally rectifiable if each closed subpath of γ is rectifiable. If γ is a rectifiable path, then γ has a parameterization by means of arc length, also called the *normal representation* of γ. The normal representation of γ is denoted by $\gamma^0\colon [0,\ell(\gamma)] \to \mathbb{R}^n$. By making use of the normal representation, one may define the integral over a locally rectifiable path γ.

Definition 4.1. Let Γ be a path family in $\mathbb{R}^n$, $n \geq 2$. Let $\mathcal{F}(\Gamma)$ be the set of all Borel functions $\rho\colon \mathbb{R}^n \to [0,\infty]$ such that

$$\int_\gamma \rho\, ds \geq 1$$

for every locally rectifiable path $\gamma \in \Gamma$. The functions in $\mathcal{F}(\Gamma)$ are called *admissible* for Γ. For $1 < p < \infty$ we define

$$\mathsf{M}_p(\Gamma) = \inf_{\rho \in \mathcal{F}(\Gamma)} \int_{\mathbb{R}^n} \rho^p\, dm \tag{4.2}$$

and call $\mathsf{M}_p(\Gamma)$ the *p-modulus* of Γ. If $\mathcal{F}(\Gamma) = \emptyset$, which is true only if Γ contains constant paths, we set $\mathsf{M}_p(\Gamma) = \infty$. The n-modulus or *conformal modulus* is denoted by $\mathsf{M}(\Gamma)$.

Lemma 4.3. [21, 6.2] *The p-modulus is an outer measure in the space of all path families in $\mathbb{R}^n$. That is,*

(1) $\mathsf{M}_p(\emptyset) = 0$,

(2) *If* $\Gamma_1 \subset \Gamma_2$ *then* $\mathsf{M}_p(\Gamma_1) \leq \mathsf{M}_p(\Gamma_2)$, *and*

(3) $\mathsf{M}_p\big(\bigcup_j \Gamma_j\big) \leq \sum_j \mathsf{M}_p\big(\Gamma_j\big)$.

Proof. (1) Since the zero function is admissible for $\emptyset$, $\mathsf{M}_p(\emptyset) = 0$.

(2) If $\Gamma_1 \subset \Gamma_2$ then $\mathcal{F}(\Gamma_2) \subset \mathcal{F}(\Gamma_1)$ and hence $\mathsf{M}_p(\Gamma_1) \leq \mathsf{M}_p(\Gamma_2)$.

(3) We may assume that $\mathsf{M}_p(\Gamma_j) < \infty$ for all j. Let $\varepsilon > 0$. Then we can choose for each j a function ρ_j admissible for Γ_j such that

$$\int_{\mathbb{R}^n} \rho_j^p\, dm \leq \mathsf{M}_p(\Gamma_j) + 2^{-j}\varepsilon.$$

Now let

$$\rho = \sup_j \rho_j, \qquad \Gamma = \bigcup_j \Gamma_j.$$

Then $\rho : \mathbb{R}^n \to [0,\infty]$ is a Borel function. Moreover, if $\gamma \in \Gamma$ is locally rectifiable, then $\gamma \in \Gamma_j$ for some j,

$$\int_\gamma \rho\, ds \geq \int_\gamma \rho_j\, ds \geq 1,$$

and hence ρ is admissible for Γ. Now

$$\mathsf{M}_p(\Gamma) \le \int_{\mathbb{R}^n} \rho^p \, dm \le \int_{\mathbb{R}^n} \sum_j \rho_j^p \, dm \le \sum_j \mathsf{M}_p(\Gamma_j) + \varepsilon.$$

By letting $\varepsilon \to 0$, the claim follows. □

Let Γ_1 and Γ_2 be path families in $\mathbb{R}^n$. We say that Γ_2 is *minorized* by Γ_1 and write $\Gamma_1 < \Gamma_2$ if every $\gamma \in \Gamma_2$ has a subpath in Γ_1.

Lemma 4.4. *If $\Gamma_1 < \Gamma_2$ then $\mathsf{M}_p(\Gamma_1) \ge \mathsf{M}_p(\Gamma_2)$.*

Proof. If $\Gamma_1 < \Gamma_2$ then obviously $\mathcal{F}(\Gamma_1) \subset \mathcal{F}(\Gamma_2)$. Hence $\mathsf{M}_p(\Gamma_1) \ge \mathsf{M}_p(\Gamma_2)$. □

Lemma 4.5. *Let G be a Borel set in $\mathbb{R}^n$, $r > 0$ and let Γ be the family of paths in G such that $\ell(\gamma) \ge r$. Then $\mathsf{M}_p(\Gamma) \le m(G) r^{-p}$.*

Proof. The claim follows immediately from (4.2) and the fact that the function $\rho = \chi_G / r$ is admissible for Γ. □

Lemma 4.6. *Path family Γ has zero p-modulus if and only if there is an admissible function $\rho \in \mathcal{F}(\Gamma)$ such that*

$$\int_{\mathbb{R}^n} \rho^p \, dm < \infty \text{ and } \int_\gamma \rho \, ds = \infty$$

for every locally rectifiable path $\gamma \in \Gamma$.

Proof. If ρ satisfies the above conditions, clearly ρ/k is admissible for Γ for all $k = 1, 2, \ldots$. Hence

$$\mathsf{M}_p(\Gamma) \le k^{-p} \int_{\mathbb{R}^n} \rho^p \, dm \to 0$$

as $k \to \infty$, and thus $\mathsf{M}_p(\Gamma) = 0$.

Now let $\mathsf{M}_p(\Gamma) = 0$ and choose a sequence of functions $\rho_k \in \mathcal{F}(\Gamma)$ such that

$$\int_{\mathbb{R}^n} \rho_k^p \, dm < 4^{-k}, \qquad k = 1, 2, \ldots.$$

Define

$$\rho(x) = \Big(\sum_{k=1}^{\infty} 2^k \rho_k^p(x) \Big)^{1/p},$$

and note that

$$\int_{\mathbb{R}^n} \rho^p \, dm < \infty.$$

On the other hand,

$$\int_\gamma \rho \, ds \ge \int_\gamma 2^{k/p} \rho_k \, ds \ge 2^{k/p} \to \infty$$

as $k \to \infty$ for every locally rectifiable path $\gamma \in \Gamma$. □

Corollary 4.7. *Let Γ be a path family in $\overline{\mathbb{R}}^n$ and denote by Γ_r the family of all rectifiable paths in Γ. Then $\mathsf{M}(\Gamma) = \mathsf{M}(\Gamma_r)$.*

The path families $\Gamma_1, \Gamma_2, \ldots$ are called *separate* if there exist disjoint Borel sets E_i such that

$$\int_\gamma \chi_{\mathbb{R}^n \setminus E_i} \, ds = 0 \tag{4.8}$$

for all locally rectifiable $\gamma \in \Gamma_i$, $i = 1, 2, \ldots$.

Lemma 4.9. [19, Proposition II.1.5] *Let $\Gamma, \Gamma_1, \Gamma_2, \ldots$ be a sequence of path families in $\mathbb{R}^n$. Then*

(1) *If $\Gamma_1, \Gamma_2, \ldots$ are separate and $\Gamma < \Gamma_j$ for all $j = 1, 2, \ldots$, then*

$$\mathsf{M}_p(\Gamma) \geq \sum_j \mathsf{M}_p(\Gamma_j).$$

Equality holds if $\Gamma = \bigcup_j \Gamma_j$.

(2) *If $\Gamma_1, \Gamma_2, \ldots$ are separate and $\Gamma_j < \Gamma$ for all $j = 1, 2, \ldots$, then*

$$\mathsf{M}_p(\Gamma)^{1/(1-p)} \geq \sum_j \mathsf{M}_p(\Gamma_j)^{1/(1-p)}, \ \ p > 1.$$

Proof. (1) Let ρ be admissible for Γ, and let E_j be as in (4.8). Then for all indices j the function $\rho_j = \chi_{E_j} \rho$ is admissible for Γ_j. It follows that

$$\sum_p \mathsf{M}_p(\Gamma_j) \leq \sum_j \int_{\mathbb{R}^n} \rho_j^p \, dm = \sum_j \int_{E_j} \rho^p \, dm \leq \int_{\mathbb{R}^n} \rho^p \, dm.$$

(2) Let E_j be as in (4.8), and let $E = \bigcup_j E_j$. Then for all indices j the function $\chi_{E_j} \rho$ is admissible for Γ_j. Let (a_j) be a sequence such that $a_j \in [0, 1]$ and $\sum_j a_j = 1$. Let

$$\rho = \sum_{j=1}^\infty a_j \chi_{E_j} \rho_j.$$

Next we show that ρ is admissible for Γ. Fix a locally rectifiable path $\gamma \in \Gamma$ and a subpath $\gamma_j \in \Gamma_j$ for each $j = 1, 2, \ldots$. Now

$$\begin{aligned}
\int_\gamma \rho \, ds &= \int_\gamma \Big(\sum_j a_j \chi_{E_j} \rho_j \Big) ds = \sum_j a_j \int_\gamma \chi_{E_j} \rho_j \, ds \\
&\geq \sum_j a_j \int_{\gamma_j} \chi_{E_j} \rho_j \, ds \geq \sum_j a_j = 1.
\end{aligned}$$

Hence ρ is admissible for Γ and

$$\begin{aligned}
\mathsf{M}_p(\Gamma) &\leq \int_{\mathbb{R}^n} \rho^p \, dm = \int_E \rho^p \, dm \\
&= \sum_j \int_{E_j} \Big(\sum_k a_k \chi_{E_k} \rho \Big)^p dm = \sum_j \int_{E_j} a_j^p \rho_j^p \, dm \\
&\leq \int_{\mathbb{R}^n} \sum_j a_j^p \rho_j^p \, dm \leq \sum_j a_j^p \int_{\mathbb{R}^n} \rho_j^p \, dm.
\end{aligned}$$

By taking the infimum over all admissible ρ_j, we obtain

$$\mathsf{M}_p(\Gamma) \le \sum_j a_j^p \mathsf{M}_p(\Gamma_j). \tag{4.10}$$

We may assume that $\mathsf{M}_p(\Gamma) > 0$ (if that would not be the case, the left side of the inequality is ∞ and there is nothing to prove). Hence by Lemma 4.4 we have $\mathsf{M}_p(\Gamma_j) \ge \mathsf{M}_p(\Gamma) > 0$. Similarly, we may assume that $\mathsf{M}_p(\Gamma_j) < \infty$.

Let

$$t_k = \frac{1}{\sum_{j=1}^k \mathsf{M}_p(\Gamma_j)^{1/(1-p)}}, \qquad a_{j,k} = \mathsf{M}_p(\Gamma_j)^{1/(1-p)} t_k,$$

for $j = 1, \dots, k$ and $k = 1, 2, \dots$. Now $\sum_{j=1}^k a_{j,k} = 1$. We choose $a_{j,k} = 0$ for $j \ge k+1$, and by (4.10) we have

$$\mathsf{M}_p(\Gamma) \le t_k^p \sum_{j=1}^k \mathsf{M}_p(\Gamma_j)^{p/(1-p)} \mathsf{M}_p(\Gamma_j) = \Big(\sum_{j=1}^k \mathsf{M}_p(\Gamma_j)^{1/(1-p)} \Big)^{1-p}.$$

By letting $k \to \infty$ the claim follows. □

For $E, F, G \subset \mathbb{R}^n$ we denote by $\Delta(E, F; G)$ the family of all nonconstant paths joining E and F in G.

Lemma 4.11. [22, 5.22] *Let $p > 1$ and let E, F be subsets of $\mathbf{H}^n$. Then*

$$\mathsf{M}_p(\Delta(E, F; \mathbf{H}^n)) \ge \frac{1}{2} \mathsf{M}_p(\Delta(E, F)).$$

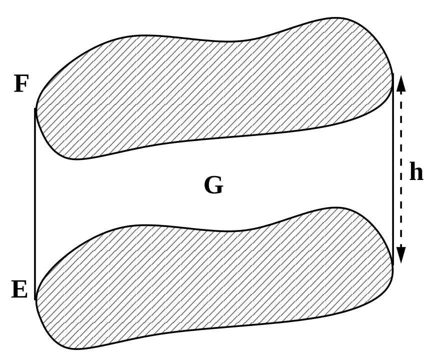

FIGURE 5: Cylinder with bases E and F.

Example 4.12. Let $E \subset \{x \in \mathbb{R}^n : x_n = 0\}$ be a Borel set, $h > 0$, $F = E + he_n$. We define a *cylinder* G with bases E, F by

$$G = \{x \in \mathbb{R}^n : (x_1, \dots, x_{n-1}, 0) \in E,\ 0 < x_n < h\}.$$

Then $\mathsf{M}_p(\Delta(E, F; G)) = m_{n-1}(E) h^{1-p} = m(G) h^{-p}$.

Proof. Choose $\rho \in \mathcal{F}(\Gamma)$ where $\Gamma = \Delta(E, F; G))$ and let γ_y be the vertical segment from $y \in E$. Then $\gamma_y \in \Gamma$. We note that $1/p + (p-1)/p = 1$, and hence by Hölder's inequality

$$1 \leq \left(\int_{\gamma_y} \rho \, ds \right)^p \leq \left(\int_{\gamma_y} 1 \, ds \right)^{p-1} \left(\int_{\gamma_y} \rho^p \, ds \right) = h^{p-1} \int_{\gamma_y} \rho^p \, ds.$$

This holds for all $y \in E$ and hence by the Fubini theorem

$$\int_{\mathbb{R}^n} \rho^p \, dm \geq \int_E \left(\int_{\gamma_y} \rho^p \, ds \right) dm_{n-1} \geq \frac{m_{n-1}(E)}{h^{p-1}}.$$

Since the above holds for any $\rho \in \mathcal{F}(\Gamma)$,

$$\mathsf{M}_p(\Gamma) \geq \frac{m_{n-1}(E)}{h^{p-1}}.$$

Next we choose $\rho = 1/h$ inside G and $\rho = 0$ otherwise. Then ρ is admissible for Γ and

$$\mathsf{M}_p(\Gamma) \leq \int_{\mathbb{R}^n} \rho^p \, dm = \frac{m_{n-1}(E)}{h^{p-1}}.$$

□

Remark 4.13. In Example 4.12 the modulus is invariant under similarity mappings if and only if $p = n$. This is the reason why the case $p = n$ is so important in the theory of quasiconformal mappings. Later in this section we will show that $\mathsf{M}(\Gamma)$ is a conformal invariant.

Ring domains. A domain G in $\overline{\mathbb{R}}^n$ is called a ring, if $\overline{\mathbb{R}}^n \setminus G$ has exactly two components. If the components are E and F, we denote the ring by $R(E, F)$.

In general, it is difficult to calculate the modulus of a given path family. Next two lemmas give us an important tool, letting us to obtain effective upper and lower bounds for the modulus in many situations.

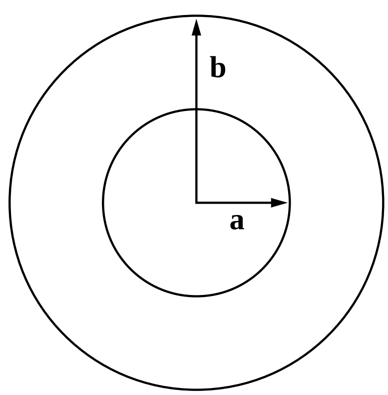

FIGURE 6: Spherical ring with $0 < a < b < \infty$.

Lemma 4.14. [21, 7.5] *Let* $0 < a < b < \infty$, $A = \mathbf{B}^n(b) \setminus \overline{\mathbf{B}}^n(a)$ *and*

$$\Gamma_A = \Delta\big(S^{n-1}(a), S^{n-1}(b); A\big).$$

Then

$$\mathsf{M}(\Gamma_A) = \omega_{n-1}\Big(\log\frac{b}{a}\Big)^{1-n}.$$

Proof. Let $\rho \in \mathcal{F}(\Gamma_A)$. For each unit vector $y \in S^{n-1}$ let $\gamma_y\colon [a,b] \to \mathbb{R}^n$ the radial line segment defined by $\gamma_y(s) = sy$. As in Example 4.12 by Hölder's inequality we obtain

$$\begin{aligned} 1 &\leq \Big(\int_{\gamma_y} \rho\, ds\Big)^n \leq \Big(\int_a^b \rho(sy)^n s^{n-1}\, ds\Big)\Big(\int_a^b \frac{1}{s}\, ds\Big)^{n-1} \\ &= \Big(\log\frac{b}{a}\Big)^{n-1}\int_a^b \rho(sy)^n s^{n-1}\, ds. \end{aligned}$$

By integrating over $y \in S^{n-1}$, we have

$$\omega_{n-1} \leq \Big(\log\frac{b}{a}\Big)^{n-1}\int_{\mathbb{R}^n} \rho^n\, dm. \tag{4.15}$$

Taking the infimum over all admissible ρ yields

$$\omega_{n-1} \leq \Big(\log\frac{b}{a}\Big)^{n-1}\mathsf{M}(\Gamma_A).$$

Next we define $\rho(x) = 1/\big(|x|\log(b/a)\big)$ for $x \in A$, and $\rho(x) = 0$ otherwise. Clearly ρ is admissible for Γ_A, and hence

$$\mathsf{M}(\Gamma_A) \leq \int_{\mathbb{R}^n} \rho^n\, dm = \omega_{n-1}\Big(\log\frac{b}{a}\Big)^{-n}\int_a^b \frac{1}{s}\, ds = \omega_{n-1}\Big(\log\frac{b}{a}\Big)^{1-n}.$$

□

Lemma 4.16. [21, 7.8] *Let* $x_0 \in \overline{\mathbb{R}}^n$ *and let* Γ *be the family of all nonconstant paths through* x_0. *Then* $\mathsf{M}(\Gamma) = 0$.

Proof. If $x_0 = \infty$, the claim follows immediately from Corollary 4.7.

If $x_0 \neq \infty$, we let

$$\Gamma_k = \{\gamma \in \Gamma : |\gamma| \cap S^{n-1}(x_0, 1/k) \neq \emptyset\}.$$

We may assume that $x_0 = 0$. Then for all $R > 1/k$

$$\Gamma_k > \Delta_R, \text{ where } \Delta_R = \Delta\big(S^{n-1}(1/k), S^{n-1}(R); \mathbf{B}^n(R) \setminus \overline{\mathbf{B}}^n(1/k)\big),$$

and by Lemma 4.4 and Lemma 4.14 we have

$$\mathsf{M}(\Gamma_k) \leq \mathsf{M}(\Delta_R) = \omega_{n-1}\Big(\log\frac{R}{1/k}\Big)^{1-n} \to 0$$

as $R \to \infty$, and thus $\mathsf{M}(\Gamma_k) = 0$. On the other hand, because $\Gamma = \bigcup_k \Gamma_k$ we have by Lemma 4.3 (3)

$$\mathsf{M}(\Gamma) \leq \sum_k \mathsf{M}(\Gamma_k) = 0.$$

□

Modulus in conformal mappings. Let $G \subset \overline{\mathbb{R}}^n$ and $f\colon G \to \overline{\mathbb{R}}^n$ be a continuous function. Suppose that Γ is a family of paths in G. Then $\Gamma' = \{f \circ \gamma : \gamma \in \Gamma\}$ is a family of paths in $f(G)$. Γ' is called the image of Γ under f.

Theorem 4.17. [21, 8.1] *If $f\colon G \to f(G)$ is conformal, then $\mathsf{M}(f(\Gamma)) = \mathsf{M}(\Gamma)$ for all path families Γ in G.*

Proof. By Lemma 4.16 we may assume that the paths of $\Gamma, f(\Gamma)$ do not go through ∞. Let $\rho_1 \in \mathcal{F}(f(\Gamma))$, and define

$$\rho(x) = \rho_1\big(f(x)\big)|f'(x)|$$

for $x \in G$ and $\rho(x) = 0$ otherwise. Because f is a conformal mapping (see [21, 5.6]),

$$\int_\gamma \rho\, ds = \int_\gamma \rho_1\big(f(x)\big)|f'(x)|\,|dx| = \int_{f\circ\gamma} \rho_1\, ds \geq 1$$

for every locally rectifiable $\gamma \in \Gamma$. It follows that $\rho \in \mathcal{F}(\Gamma)$, and

$$\mathsf{M}(\Gamma) \leq \int_{\mathbb{R}^n} \rho^n\, dm = \int_G \rho_1^n\big(f(x)\big)|J_f(x)|\, dm = \int_{f(G)} \rho_1^n\, dm = \int_{\mathbb{R}^n} \rho_1^n\, dm$$

for all $\rho_1 \in \mathcal{F}(f(\Gamma))$, and thus $\mathsf{M}(\Gamma) \leq \mathsf{M}(f(\Gamma))$. The inverse inequality follows from the fact that f^{-1} is conformal. □

Lemma 4.18. *Let $A \subset \mathbf{H}^n$, $B \subset (\complement\mathbf{H}^n)$, $\Gamma = \Delta(A,B)$, and let*

$$\Gamma_1 = \Delta(A, \partial\mathbf{H}^n),\ \ \Gamma_2 = \Delta(B, \partial\mathbf{H}^n).$$

Then

$$\mathsf{M}(\Gamma) \leq 2^{-n}\big(\mathsf{M}(\Gamma_1) + \mathsf{M}(\Gamma_2)\big).$$

In particular, the equality holds if $A = g(B)$, where g is the reflection in $\mathbf{H}^n$.

Proof. Let $\rho_1 \in \mathcal{F}(\Gamma_1)$ and $\rho_2 \in \mathcal{F}(\Gamma_2)$. We note that if $\gamma \in \Gamma$ is a rectifiable path, then γ has subpaths γ_1, γ_2 such that $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$. Thus

$$1 \leq \frac{1}{2}\int_{\gamma_1} \rho_1\, ds + \frac{1}{2}\int_{\gamma_2} \rho_2\, ds.$$

We define $\rho = \rho_1/2 + \rho_2/2$. Now ρ is an admissible function for the curve family Γ and hence

$$\mathsf{M}(\Gamma) \leq \int_{\mathbb{R}^n} \rho^n\, dm.$$

We may assume that $\rho_1(z) = 0$ for $z \notin \mathbf{H}^n$, and $\rho_2(z) = 0$ for $z \in \mathbf{H}^n$. As $\rho = \rho_1/2 + \rho_2/2$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \rho^n\, dm &= 2^{-n}\int_{\mathbf{H}^n} \rho_1^n\, dm + 2^{-n}\int_{\mathbb{R}^n\setminus\mathbf{H}^n} \rho_2^n\, dm \\ &= 2^{-n}\Big(\int_{\mathbb{R}^n} \rho_1^n\, dm + \int_{\mathbb{R}^n} \rho_2^n\, dm\Big).\end{aligned}$$

It follows that

$$\mathsf{M}(\Gamma) \le 2^{-n}\big(\mathsf{M}(\Gamma_1) + \mathsf{M}(\Gamma_2)\big).$$

Next we consider the case $A = g(B)$. Let ρ be an admissible function for the path family Γ and denote $\rho \circ g$ by $\bar\rho$. Now the function

$$\hat\rho = \begin{cases} \rho + \bar\rho & \text{on } \mathbf{H}^n, \\ 0 & \text{on } \complement\mathbf{H}^n, \end{cases}$$

is admissible for the path family Γ_1. By the inequality $(a+b)^n \le 2^{n-1}(a^n + b^n)$ (for $a, b \ge 0$) and the fact that $\mathsf{M}(\Gamma_1) = \mathsf{M}(\Gamma_2)$ it follows that

$$\begin{aligned} \mathsf{M}(\Gamma_1) &\le \int_{\mathbb{R}^n} \hat\rho^n dm = \frac{1}{2}\int_{\mathbb{R}^n} (\rho + \bar\rho)^n dm \\ &\le 2^{n-2}\int_{\mathbb{R}^n} (\rho^n + \bar\rho^n) dm = 2^{n-1}\int_{\mathbb{R}^n} \rho^n dm. \end{aligned}$$

Hence,

$$\mathsf{M}(\Gamma_1) + \mathsf{M}(\Gamma_2) = 2\mathsf{M}(\Gamma_1) \le 2^n \int_{\mathbb{R}^n} \rho^n dm,$$

for any ρ admissible for the curve family Γ. By taking infimum over all admissible ρ, the claim follows. □

Capacity of a condenser. A *condenser* in $\mathbb{R}^n$ is a pair $E = (A, C)$, where A is open in $\mathbb{R}^n$ and C is a compact subset of A. The p-*capacity* of E is defined by

$$\text{(4.19)} \qquad \mathrm{cap}_p E = \inf_u \int_A |\nabla u|^p dm, \;\; 1 \le p < \infty,$$

where the infimum is taken over all nonnegative functions u in $\mathrm{ACL}^p(A)$ with compact support in A and $u|C \ge 1$. The n-capacity of E is called the *conformal capacity* of E and denoted by $\mathrm{cap}E$.

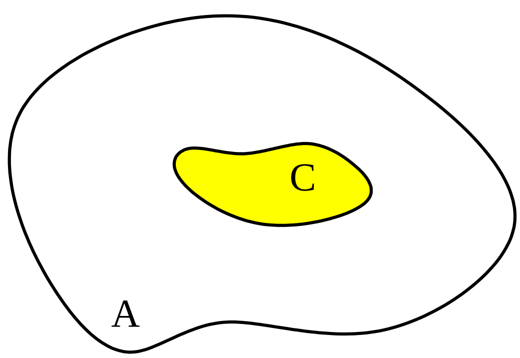

FIGURE 7: Condenser $E = (A, C)$.

Lemma 4.20. [22, 7.9] *For all condensers* (A, C) *in* $\mathbb{R}^n$

$$\text{(4.21)} \qquad \mathrm{cap}(A, C) = \mathsf{M}\big(\Delta(C, \partial A; A)\big).$$

Sets of zero capacity. A compact set E in $\mathbb{R}^n$ is said to be of *capacity zero*, denoted cap$E = 0$, if there exists a bounded set A with $E \subset A$ and $\mathrm{cap}(A, E) = 0$. A compact set $E \subset \overline{\mathbb{R}}^n$, $E \neq \overline{\mathbb{R}}^n$ is said to be of capacity zero if E can be mapped by a Möbius transformation onto a bounded set of capacity zero. Otherwise E is said to be of positive capacity, and we write cap$E > 0$.

Spherical symmetrizations. Let L be a ray from x_0 to ∞ and $E \subset \overline{\mathbb{R}}^n$ be a compact set. We define *spherical symmetrization* of E in L as the set E^* satisfying the following conditions:

(1) $x_0 \in E^*$ if and only if $x_0 \in E$,
(2) $\infty \in E^*$ if and only if $\infty \in E$,
(3) For $r \in (0, \infty)$ the set $E^* \cap S^{n-1}(x_0, r)$ is a closed spherical cap centered on L with the same $(n-1)$-dimensional Lebesgue measure as $E \cap S^{n-1}(x_0, r)$ for $E \cap S^{n-1}(x_0, r) \neq \emptyset$ and $\emptyset$ otherwise.

We note that E^* is always compact and connected if E is.

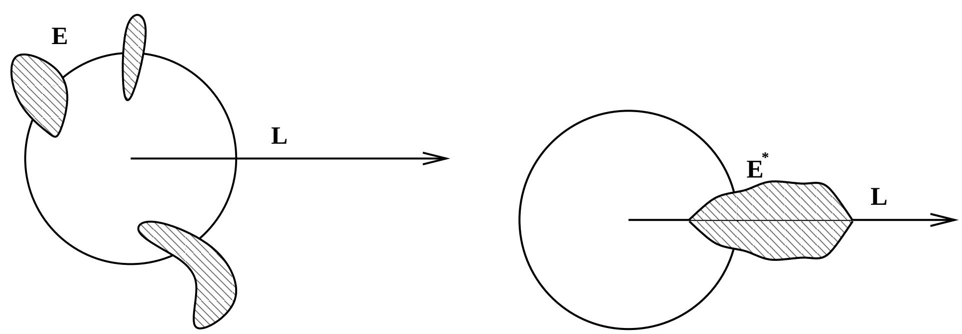

FIGURE 8: Spherical symmetrization.

Theorem 4.22. *If E^* is the spherical symmetrization of E in a ray L, then*

(1) $m(E^*) = m(E)$, *and*
(2) $m_{n-1}(\partial E^*) \leq m_{n-1}(\partial E)$.

Proof. (Outline, [7, p.224]) By Fubini's theorem

$$m(E^*) = \int_0^\infty m_{n-1}(E^* \cap S^{n-1}(x_0, r))dr = \int_0^\infty m_{n-1}(E \cap S^{n-1}(x_0, r))dr = m(E),$$

which gives the first part.

To prove the second part, assume first that E is a polyhedron. Then for $r \in (0, \infty)$ the Brunn–Minkowski inequality yields

$$E^*(r) = \{x : d(x, E^*) \leq r\} \subset \{x : d(x, E) \leq r\}^* = E(r)^*,$$

and hence

$$\begin{aligned} m_{n-1}(\partial E^*) &\leq \limsup_{r\to 0} \frac{m(E^*(r)) - m(E^*)}{2r} \\ &\leq \limsup_{r\to 0} \frac{m(E(r)) - m(E)}{2r} = m_{n-1}(\partial E). \end{aligned}$$

The result for the general domains is obtained by approximating the boundary with polyhedrons. □

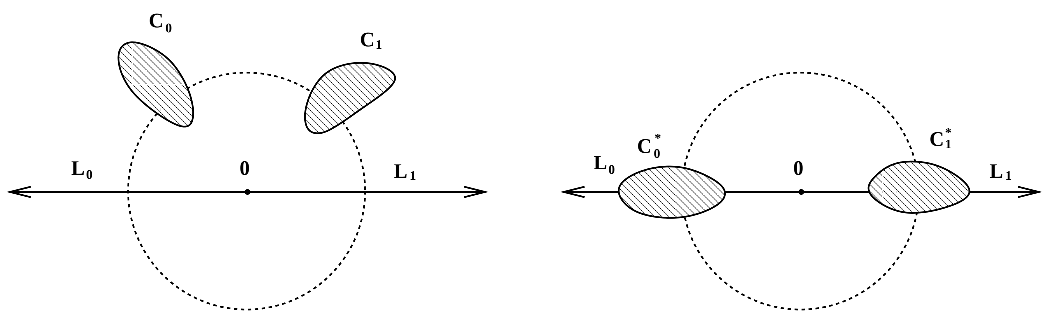

FIGURE 9: Spherical symmetrization of a ring.

Theorem 4.23. *If $R = R(C_0, C_1)$ is a ring and if C_0^* and C_1^* are the sphrerical symmetrizations of C_0 and C_1 in opposite rays L_0, L_1, then $R^* = R(C_0^*, C_1^*)$ is a ring with* $\operatorname{cap} R^* \leq \operatorname{cap} R$.

Proof. (Idea, [7, p.225]) Let u be a locally lipschitz function that is admissible for R. Choose u^* such that $\{x : u^*(x) \leq t\} = \{x : u(x) \leq t\}^*$. Then u^* is admissible for R^* and from Theorem 4.22 we obtain

$$\operatorname{cap}(R^*) \leq \int_{\mathbb{R}^n} |\nabla u^*|^n dm \leq \int_{\mathbb{R}^n} |\nabla u|^n dm.$$

By taking the infimum over all admissible u the claim follows. □

Canonical ring domains. The complementary components of the *Grötzsch ring* $R_{G,n}(s)$ in $\mathbb{R}^n$ are $\overline{\mathbf{B}}^n$ and $[se_1, \infty]$, $s > 1$, and those of the *Teichmüller ring* $R_{T,n}(s)$ are $[-e_1, 0]$ and $[se_1, \infty]$, $s > 0$. We define two special functions $\gamma_n(s)$, $s > 1$ and $\tau_n(s)$, $s > 0$ by

$$\begin{cases} \gamma_n(s) = \mathsf{M}\big(\Delta(\overline{\mathbf{B}}^n, [se_1, \infty])\big) = \gamma(s), \\ \tau_n(s) = \mathsf{M}\big(\Delta([-e_1, 0], [se_1, \infty])\big) = \tau(s), \end{cases}$$

respectively. The subscript n is omitted if there is no danger of confusion. We shall refer to these functions as the *Grötzsch capacity* and the *Teichmüller capacity*.

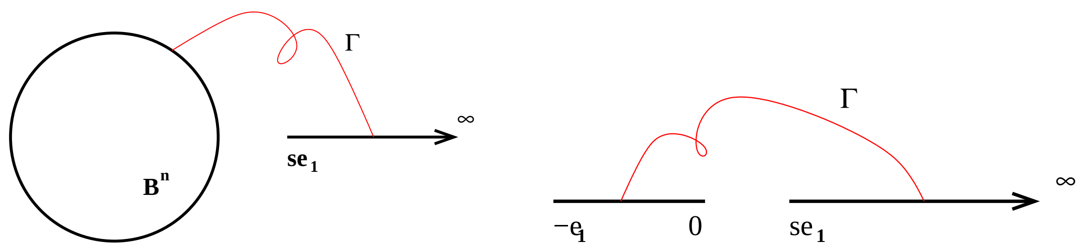

FIGURE 10: Grötzsch ring $R_{G,n}(s)$ (left) and Teichmüller ring $R_{T,n}(s)$ (right).

Lemma 4.24. [22, 5.53] *For all* $s > 1$

$$\gamma_n(s) = 2^{n-1}\tau_n(s^2 - 1)$$

and that $\tau_n \colon (0,\infty) \to (0,\infty)$ *is a decreasing homeomorphism.*

Proof. (Idea) Apply Lemma 4.18 and an auxiliary Möbius transformation. □

Lemma 4.25. [22, 5.63(1)] *Let* $s > 0$. *Then*

$$\tau(s) \le \gamma(1+2s) = 2^{n-1}\tau(4s^2 + 4s)$$

Proof. Let $\Gamma = \Delta(S^{n-1}(-e_1/2, 1/2), [se_1, \infty])$. Then by Lemma 4.24

$$\mathsf{M}(\Gamma) = \gamma(1+2s) = 2^{n-1}\tau(4s^2 + 4s).$$

By Lemma 4.4 $\tau(s) \le \mathsf{M}(\Gamma)$. □

Lemma 4.26. [3, (8.65),(8.62)] *The following estimates hold for* $\tau_n(t)$, $t > 0$*:*

$$\tau_n(t) \ge 2^{1-n}\omega_{n-1}\bigg(\log\Big(\frac{\lambda_n}{2}(\sqrt{1+t} + \sqrt{t})\Big)\bigg)^{1-n},$$

and for $\gamma_n(1/r)$, $r \in (0,1)$*:*

$$\gamma_n(1/r) \ge \omega_{n-1}\bigg(\log\frac{\lambda_n\big(1+\sqrt{1-r^2}\big)}{2r}\bigg)^{1-n} \ge \omega_{n-1}\Big(\log\frac{\lambda_n}{r}\Big)^{1-n},$$

where λ_n *is the Grötzsch ring constant depending only on* n.

The value of λ_n is known only for $n = 2$, namely $\lambda_2 = 4$. For $n \ge 3$ it is known that $2^{0.76(n-1)} \le \lambda_n \le 2e^{n-1}$. For more information on λ_n, see [3, p.169].

Lemma 4.27. *(see* [9, 2.31]*) Let* $0 < r_0 < 1$. *Then*

$$\mathsf{M}\big(\Delta(\mathbf{B}^n(r), S^{n-1})\big) \ge \gamma_n(1/r) \ge C(n, r_0)\mathsf{M}\big(\Delta(\mathbf{B}^n(r), S^{n-1})\big),$$

for $r_0 > r > 0$.

Proof. By Lemma 4.26,

$$\gamma_n(1/r) \ge \omega_{n-1}\bigg(\log\frac{\lambda_n\big(1+\sqrt{1-r^2}\big)}{2r}\bigg)^{1-n} \ge \omega_{n-1}\Big(\log\frac{\lambda_n}{r}\Big)^{1-n}.$$

We note that

$$\log \frac{\lambda_n}{r} \le \Big(1 - \frac{\log \lambda_n}{\log r_0}\Big)\Big(\log \frac{1}{r}\Big),$$

for $0 < r < r_0$. Thus

$$\begin{aligned}\gamma_n(1/r) &\ge C(n,r_0)\omega_{n-1}\Big(\log \frac{1}{r}\Big)^{1-n} \\ &= C(n,r_0)\mathsf{M}\big(\Delta(\mathbf{B}^n(r), S^{n-1})\big),\end{aligned}$$

with

$$C(n,r_0) = \Big(1 - \frac{\log \lambda_n}{\log r_0}\Big)^{1-n}.$$

The second inequality follows immediately from the fact that the line segment $[0,r)$ is contained in the ball of radius r. □

Remark 4.28. Note that $C(n,r_0) \to 1$ as $r_0 \to 0$ in Lemma 4.27.

Lemma 4.29. [22, 7.34] *Let $R = R(E,F)$ be a ring in $\mathbb{R}^n$, and let $a,b \in E$, $c,\infty \in F$ be distinct points. Then*

$$\mathsf{M}(\Delta(E,F)) \ge \tau\Big(\frac{|a-c|}{|a-b|}\Big).$$

Equality holds for $E = [-e_1, 0]$, $a = 0$, $b = -e_1$, $F = [se_1, \infty)$, $c = se_1$, $d = \infty$.

It is not obvious from the definition how $\mathsf{M}(\Delta(E,F))$, for nonempty $E,F \in \overline{\mathbb{R}}^n$, depends on the geometric setup and the structure of the sets E,F. The following lemma gives a lower bound for $\mathsf{M}(\Delta(E,F))$ in the terms of $d(E,F)/\min\{d(E),d(F)\}$.

Lemma 4.30. [22, 7.38] *Let E,F be disjoint continua in $\mathbb{R}^n$ with $d(E),d(F) > 0$. Then*

$$\mathsf{M}(\Delta(E,F)) \ge \tau(4s^2 + 4s) \ge c_n \log(1 + 1/s)$$

where $s = d(E,F)/\min\{d(E),d(F)\}$ and $c_n > 0$ is a constant depending only on n.

This result can be improved to the following Lemma, which shows that $\mathsf{M}(\Delta(E,F))$ and $s = d(E,F)/\min\{d(E),d(F)\}$ are simultaneously small or large, provided that E,F are connected.

Lemma 4.31. [9, 2.30] *For $n \ge 2$ there are homeomorphisms h_1, h_2 of the positive real axis with the following property. If E,F are the components of the complements of a nondegenerate ring domain in $\overline{\mathbb{R}}^n$, then*

$$h_1(s) \le \mathsf{M}(\Delta(E,F)) \le h_2(s),$$

where $s = d(E,F)/\min\{d(E),d(F)\}$.

Spherical metric. The *stereographic projection* $\pi\colon \overline{\mathbb{R}}^n \to S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ is defined by

$$\pi(x) = e_{n+1} + \frac{x - e_{n+1}}{|x - e_{n+1}|^2}, \;\; x \in \mathbb{R}^n; \;\; \pi(\infty) = e_{n+1}. \tag{4.32}$$

Stereographic projection is the restriction to $\overline{\mathbb{R}}^n$ of the inversion in $S^n(e_{n+1}, 1)$ in $\overline{\mathbb{R}}^{n+1}$. Since $\pi^{-1} = \pi$, it follows that π maps the *Riemann sphere* $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ onto $\overline{\mathbb{R}}^n$. The *chordal metric* q in $\overline{\mathbb{R}}^n$ is defined by

$$q(x, y) = |\pi(x) - \pi(y)|; \;\; x, y \in \overline{\mathbb{R}}^n. \tag{4.33}$$

Lemma 4.34. [22, 7.37] *If $R = R(E, F)$ is a ring, then*

$$\mathsf{M}(\Delta(E, F)) \;\; \geq \;\; \tau\left(\frac{1}{q(E)q(F)}\right), \tag{4.35}$$

$$\mathsf{M}(\Delta(E, F)) \;\; \geq \;\; \tau\left(\frac{4q(E, F)}{q(E)q(F)}\right). \tag{4.36}$$

5. Quasiconformal mappings

A homeomorphism $f\colon G \to \mathbb{R}^n$, $n \geq 2$, of a domain G in $\mathbb{R}^n$ is called *quasiconformal* if f is in ACL^n, and there exists a constant K, $1 \leq K < \infty$ such that

$$|f'(x)|^n \leq K|J_f(x)|, \;\; |f'(x)| = \max_{|h|=1} |f'(x)h|,$$

a.e. in G, where $f'(x)$ is the formal derivative. The smallest $K \geq 1$ for which this inequality is true is called the outer dilatation of f and denoted by $K_O(f)$. If f is quasiconformal, then the smallest $K \geq 1$ for which the inequality

$$|J_f(x)| \leq Kl(f'(x))^n, \;\; l(f'(x)) = \min_{|h|=1} |f'(x)h|,$$

holds a.e. in G is called the inner dilatation of f and denoted by $K_I(f)$. The maximal dilatation of f is the number $K(f) = \max\{K_I(f), K_O(f)\}$. If $K(f) \leq K$, f is said to be K-quasiconformal. It is well-known that

$$K_I(f) \leq K_O^{n-1}(f), \qquad K_O(f) \leq K_I^{n-1}(f),$$

and hence $K_I(f)$ and $K_O(f)$ are simultaneously finite.

Theorem 5.1. [21, 32.2,33.2] *Let $f\colon G \to \mathbb{R}^n$ be a quasiconformal mapping. Then*

(1) *f is differentiable a.e.,*
(2) *f satisfies condition (N), i.e. if $A \subset G$ and $m(A) = 0$, then $m(fA) = 0$.*

The next lemma gives another definition of quasiconformality. This definition is called the geometric definition, and it is very useful in applications. The proof for equivalence of these definitions is given in [21].

Lemma 5.2. *A homeomorphism $f\colon G \to G'$ is K-quasiconformal if and only if*

$$\mathsf{M}(\Gamma)/K \le \mathsf{M}(f(\Gamma)) \le K\mathsf{M}(\Gamma)$$

for every path family Γ in G.

We may also give geometric definitions for the inner and outer dilatations. Again, we refer to [21] for the proofs for the equivalence of these definitions.

Let G, G' be domains in $\overline{\mathbb{R}}^n$ and $f\colon G \to G'$ be a homeomorphism. Then inner and outer dilatations of f are respectively

$$K_I(f) = \sup \frac{M(f(\Gamma))}{M(\Gamma)}, \quad K_O(f) = \sup \frac{M(\Gamma)}{M(f(\Gamma))},$$

where the suprema are taken over all path families Γ in G such that $M(\Gamma)$ and $M(f(\Gamma))$ are not simultaneously 0 or ∞. The maximal dilatation of f is

$$K(f) = \max\{K_I(f), K_O(f)\}.$$

Theorem 5.3. [21, 13.2] *Let $f\colon G' \to G''$, $g\colon G \to G'$ be quasiconformal mappings. Then*

(1) $K_I(f^{-1}) = K_O(f)$,
(2) $K_O(f^{-1}) = K_I(f)$,
(3) $K(f^{-1}) = K(f)$,
(4) $K_I(f \circ g) \le K_I(f)K_I(g)$,
(5) $K_O(f \circ g) \le K_O(f)K_O(g)$,
(6) $K(f \circ g) \le K(f)K(g)$.

Examples. (see [21, pp.49-50]) (1) A homeomorphism $f\colon G \to fG$ satisfying

$$|x-y|/L \le |f(x) - f(y)| \le L|x-y|$$

for all $x, y \in G$ is called L-bilipschitz. It is easy to see that L-bilipschitz maps are $L^{2(n-1)}$ -quasiconformal.

(2) Let $a \ne 0$ be a real number, and let $f(x) = |x|^{a-1}x$. We can extend f to a homeomorphism $f\colon \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ by defining $f(0) = 0$, $f(\infty) = \infty$ for $a > 0$ and $f(0) = \infty$, $f(\infty) = 0$ for $a < 0$. Then f is quasiconformal with

$$\begin{array}{lllllll} K_I(f) & = & |a|, & K_O(f) & = & |a|^{n-1} & \text{if } |a| \ge 1, \\ K_I(f) & = & |a|^{1-n}, & K_O(f) & = & |a|^{-1} & \text{if } |a| \le 1. \end{array}$$

(3) Let (r, φ, z) be the cylindrical coordinates of a point $x \in \mathbb{R}^n$, i.e. $r \ge 0$, $0 \le \varphi \le 2\pi$, $z \in \mathbb{R}^{n-2}$, and

$$\left\{ \begin{array}{lll} x_1 & = & r\cos\varphi, \\ x_2 & = & r\sin\varphi, \\ x_j & = & z_{j-2} \text{ for } 3 \le j \le n. \end{array} \right.$$

The domain G_α, defined by $0 < \varphi < \alpha$, is called a wedge of angle α, $\alpha \in (0, 2\pi)$. Let $0 < \alpha \le \beta < 2\pi$. The folding $f\colon G_\alpha \to G_\beta$, defined by

$$f(r, \varphi, z) = (r, \beta\varphi/\alpha, z),$$

is quasiconformal with $K_I(f) = \beta/\alpha$, $K_O(f) = (\beta/\alpha)^{n-1}$.

6. An application of the modulus technique

As an application, we give a bound for how close to a point α the values attained by a quasiconformal mapping on a sequence of continua approaching the boundary can be. The bound is given in the terms of the diameter of the continua involved. In order to prove this result, we need the following lemmas. This result is presented in [17, pp.638–639].

Lemma 6.1. *Let $w > 0$ and $t \in (0, \min\{w^2, 1/w\})$. Then*

$$\frac{1}{2}\log\frac{1}{t} < \log\frac{w}{t} < 2\log\frac{1}{t}.$$

Proof. Since $t < w^2$, we have $1/\sqrt{t} < w/t$. On the other hand, $t < 1/w$, or $w < 1/t$, and hence $w/t < 1/t^2$. By taking logarithm the claim follows. $\square$

Lemma 6.2. *Let $C \subset \mathbf{B}^n$ be connected and $0 < d(C) \le 1$. Then $m \equiv d(0,C)/d(C) < \infty$ and if $m > 0$, then*

$$\mathsf{M}(\Gamma) \ge \frac{1}{2}\tau(4m^2 + 4m) \ge 2^{-n}\tau(m);\;\; \Gamma = \Delta(\overline{\mathbf{B}}^n(1/2), C; \mathbf{B}^n).$$

Proof. The second inequality holds by Lemma 4.25. To prove the first inequality, we note that if $C \cap \overline{\mathbf{B}}^n(1/2) \ne \emptyset$, then $\mathsf{M}(\Gamma) = \infty$ and there is nothing to prove. In what follows we may assume that $C \cap \overline{\mathbf{B}}^n(1/2) = \emptyset$. Now the result follows from the symmetry property of the modulus Lemma 4.11 and Lemma 4.30. $\square$

Theorem 6.3. *Let $f\colon \mathbf{B}^n \to \mathbb{R}^n$ be a quasiconformal mapping or constant, $\alpha \in \mathbb{R}^n$ and C_j a sequence of nondegenerate continua such that $C_j \to \partial\mathbf{B}^n$ and $|f(x) - \alpha| < M_j$ when $x \in C_j$, where $M_j \to 0$ as $j \to \infty$. If*

$$\limsup_{j\to\infty} \tau\left(\frac{1}{d(C_j)}\right)\left(\log\frac{1}{M_j}\right)^{n-1} = \infty,$$

then $f \equiv \alpha$. In particular, if

$$\limsup_{j\to\infty} \left(\log\frac{1}{d(C_j)}\right)^{1-n}\left(\log\frac{1}{M_j}\right)^{n-1} = \infty,$$

then $f \equiv \alpha$.

Proof. Suppose that f is not constant. Let $\Gamma_j = \Delta(\mathbf{B}^n(1/2), C_j; \mathbf{B}^n)$. Then by Lemma 6.2

$$\mathsf{M}(\Gamma_j) \ge 2^{-n}\tau\left(\frac{d(0,C_j)}{d(C_j)}\right) \ge 2^{-n}\tau\left(\frac{1}{d(C_j)}\right).$$

Let $w = d(f\overline{\mathbf{B}}^n(1/2), \alpha) > 0$. Now by Lemma 4.14

$$\mathsf{M}(f\Gamma_j) \le \omega_{n-1}\left(\log\frac{w}{M_j}\right)^{1-n} \le \omega_{n-1}\left(\frac{1}{2}\log\frac{1}{M_j}\right)^{1-n},$$

whenever $M_j < \min\{w^2, 1/w\}$ by Lemma 6.1. Because $\mathsf{M}(\Gamma_j) \le K(\mathsf{M}(f\Gamma_j))$, the estimates above yield

$$\tau\left(\frac{1}{d(C_j)}\right)\left(\log\frac{1}{M_j}\right)^{n-1} \quad\le\quad 2^{2n-1}K\omega_{n-1},$$

proving the first part of the claim.

The estimate (4.26) yields

$$\tau(t) \ge 2^{1-n}\omega_{n-1}\Big[\log\Big(\frac{\lambda_n}{2}\big(\sqrt{1+t}+\sqrt{t}\big)\Big)\Big]^{1-n}$$

where $t = 1/d(C_j)$. It follows that

$$\begin{aligned}\Big[\log\Big(\frac{\lambda_n}{2}\big(\sqrt{1+t}+\sqrt{t}\big)\Big)\Big]^{1-n} \quad&\ge\quad \Big[\log\Big(\frac{\lambda_n}{2}\big(1+2\sqrt{t}\big)\Big)\Big]^{1-n}\\ &=\quad \Big[\log\Big(\frac{\lambda_n}{2}\big(1+\frac{2}{\sqrt{d(C_j)}}\big)\Big)\Big]^{1-n}.\end{aligned}$$

We note that

$$\Big[\log\Big(\frac{\lambda_n}{2}\big(1+\frac{2}{\sqrt{d(C_j)}}\big)\Big)\Big]^{1-n} \ge \Big[2\log\big(\frac{\lambda_n}{\sqrt{d(C_j)}}\big)\Big]^{1-n}$$

whenever j is large enough. Let $v = \lambda_n$. Now by Lemma 6.1

$$\Big[2\log\big(\frac{\lambda_n}{\sqrt{d(C_j)}}\big)\Big]^{1-n} \ge \left(2\log\frac{1}{d(C_j)}\right)^{1-n},$$

for $\sqrt{d(C_j)} < \min\{v^2, 1/v\}$. Hence

$$\tau\left(\frac{1}{d(C_j)}\right)\left(\log\frac{1}{M_j}\right)^{n-1} \le 2^{2-2n}\omega_{n-1}\left(\log\frac{1}{d(C_j)}\right)^{1-n}\left(\log\frac{1}{M_j}\right)^{n-1},$$

which gives the second part of the claim. □

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