MAT 331 Fall 2017, Project 5 (modified)
The Weierstrass approximation theorem

The Weierstrass theorem says that any continuous function on a closed, bounded interval can be uniformly approximated by polynomials. This project explores several different proofs of this well known theorem that are discussed in the exercises of Chapter 6 of Trefethen's book "Approximation Theory and Approximation Practice". The first six chapters of this book are available for download via a link on the MAT 331 webpage.

(1) Give a careful statement of the theorem. Explain what uniform approximation means.

(2) Look up Weierstrass and give a brief historical desciption of him, e.g., where and when did he live, what is he known for?

(3) Plot $f(x) = |x|^{1/2}\sin(1/x)$ and its Chebfun approximations for several values of n (the degree of the interpolating polynomial). How fast does the maximum difference go to zero as n increases?

(4) Describe the proof of Weierstrass' theorem sketeched in the book that involves convolution with a Gaussian. For the function above, plot f and its convolution with several Guassians for different values of t. Plot the maximum difference between f and the convolutions as a function of t.

(5) Write `MATLAB` code to implement Bernstein's proof given in Exercise 6.4. Apply this code to the function f above and plot the function with its approximations, and plot how the maximum error goes down with n. Prove (as asked in the problem) that $\min f \leq B_n(x) \leq \max f$. Does this appear to hold in your plots?

This version is modified from the original by removing part (6).