MAT 331 Fall 2017, Project 3
Uniformly discrete forests with poor visibility

This project is based on the 2017 preprint " Uniformly discrete forests with poor visibility" by Noga Alon.

Consider a set of points in the plane so that any two points are at least distance 1 apart. Put a disk of radius $r < 1/2$ around each point. If L is large, must very line segment of length L hit at least one disk. If the points are grid points, the answer is no; you can place the segment between rows of disks. But there are examples where the answer is yes. These are called dense forests, because if we think if the disks as cross-section of tree trunks, then we can never see farther than distance L in this forest.

This paper gives the best known construction of a dense forest (it gives the smallest known L for each r). It uses a random construction and shows the result set works with positive probability. The project is to test this numerically. The constriction is just the top half of page 3; the rest of Section 2 is the proof that it works, but you don't need to understand this to test the construction.

(1) Section 2 of the paper defines a cross using a union of five unit squares. Write a function that will choose a random point in a cross. (Choose one of the five squares randomly, then choose point randomly in that square by choosing the x and y coordinated uniformly at random.)
(2) Write a function that takes a point inside the cross and computes the distance to the boundary.
(3) Write a function that will choose a random point in the cross that is at least a given least distance d from the boundary of the cross. (You can do this by repeated applying step 1 until you get such a point). d should be $0 < d < 1/4$.
(4) The plane can be tiled by such crosses. Find the centers of the crosses that are needed to cover a large box, say $[-n, n] \times [n, n]$ for n large, say 100. They from a lattice generated by two vectors, so the centers are integer combinations of these. Draw a picture of the tiling.
(5) Following Alon's paper, choose one random point in each cross in the tiling that is at least distance d from the boundary. Add the points to your drawing of the tiling.
(6) Test the claim of the paper is that this set will be a dense forest with high probability. Choose random line segment of length L and compute the distance to closest point. Do this for many random choices and scatter plot the results for different values of r. The longer L gets, the smaller the distance should get. Does this happen? For each value of L do many random experiments and compute the maximum and average over the experiments.
(7) Repeat the whole construction several times. Does it always work, or does it fail sometimes? The paper only claims that this procedure works most of the time.