MAT 331 Fall 2017, Project 2
Points of increase for random walk

This project is based on the paper "Points of increase for Brownian motion" by Yuval Peres. A copy of this paper is available on the MAT 331 website.

Consider at random walk on the integers that moves left or right at each step with equal probability. Let X_k denote the position of the random walk at step k (assume the walk starts at $X_0 = 0$). Suppose the walk has N steps. We say n, $1 \leq n \leq N$ is a point of increase if $X_k \leq X_n$ for all $0 \leq k \leq n$ and $X_k \geq X_n$ for all $n \leq k \leq N$.

(1) Fix a large number, say $N = 1000$ and write a `MATLAB` script that creates a 1000 long random vector with values in ± 1, and use the command `cumsum` to create the random walk `x`. Then write a function that takes this vector as input and checks whether this random walk has a point of increase or not.

(2) Choose a large M, say $M = 10,000$, and repeat the experiment M times. How many times does the random walk have a point of increase?

(3) Peres' paper gives an upper and lower bound for the probability that there is a point of increase in terms of quantities

$$p_n = \text{ probability that } X_k \geq 0 \text{ for all } 1 \leq k \leq n.$$

The precise values for these are given by

$$p_n = \binom{n}{n/2} 2^{-n}, \qquad n \text{ even },$$

$$p_n = \binom{n}{(n-1)/2} 2^{-n}, \quad n \text{ odd },$$

which you may use without proof (a proof is given in Section 6.11 of my book with Peres *Fractals in probability and analysis* which can be found on my homepage. Using the formulas in the paper, compute the lower and upper bound for the case you did in the experiment. For values of n up to around 1000, you can use the `nchoosek` command to compute the binomial coefficients, but you will get messages about lack of accuracy (these won't effect your answer much). Using the formulas above, it is not hard to derive a formula for p_{n+2} in terms of p_n, and using this will give a better result for much larger values of n.

(4) How does your experimental result compare to the theoretical predictions?

(5) Repeat the experiment for some other values of N and M and present the results using tables or plots or both. In general, is Peres' upper or lower bound closer to the experimental values?

Optional follow-up project: At the end of this paper, Peres states a conjecture. A possible follow-up project is to write `MATLAB` code to test this conjecture. If you are interested in doing this, come talk to me.