MAT 331 Fall 2017, Project 11
Fourier series and Gibbs phenomenon

Every 2π-periodic integrable function f has a Fourier expansion given by

$$\sum_{n=-\infty}^{\infty} c_n \exp(int), \text{ where } c_n = \frac{1}{2pi}\int_{-\pi}^{\pi} f(t)\exp(-int)dt.$$

This is the complex notation where $\exp(it) = \cos(t) + i\sin(t)$. For real-valued functions, we some times use the equivalent expansion

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(t),$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt.$$

You may use either form for this project (they should give the same results).

(1) Suppose f is 2π periodic and $f(x) = x$ on the interval $(-\pi, \pi]$. Use the definitions to derive that the Fourier series of f is given by $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(nt)$. How can you express f in `MATLAB` so that it gives the correct value for all real x? Remember that f has jump discontinuities at all the points $\pi + 2\pi n$, for n an integer. Plot f on the interval $[-2\pi, 2\pi]$.

(2) For $n = 20, 50, 100$ plot f and the nth partial sum of the Fourier series on the same graph. Let the horizontal axis range from -2π to 2π, so that the discontinuity of f is visible.

(3) The maximum value of f is π. Make a table of the maximum values of the nth partial sums for $n = 10, 20, \ldots, 100$. Do these appear to be approaching a limiting value? It should appear that the Fourier partial sums "overshoot" the graph of f near the jump discontinuity. This is called the "Gibbs phenomenon". Estimate this value (use larger n, if possible).

(4) Theory predicts that the partial sums "overshoot" the value of f at the jump discontinuity by $(A-1)\pi$ where

$$A = \frac{2}{\pi}\int_0^{\pi} \frac{\sin(x)}{x} dx.$$

Estimate this integral numerically. How does this prediction compare to the amount of "overshoot" that you observed in the previous part?

(5) The Cesàro sum of a finite Fourier series is

$$a_0 + \sum_{n=1}^{N}(1 - \frac{n}{N})a_n \cos(nt) + b_n \sin(t).$$

Usually this sum is better behaved than the original partial Fourier sum. Repeat the plots in part (3), but now also add the plot of the Cesàro sum as well. Does it exhibit the Gibbs phenomenon as well? Which series gives a better approximation to f?