PROBLEM SET 6 - Product measures

1. Suppose $E \subset R^2$ is such that $E_y = \{x : (x, y) \in E\} \subset R$ has measure zero for almost every y. Show that E has measure zero.

2. If f and g are measurable on R, show that $h(x, y) = f(x)g(y)$ is measurable on R^2.

3. Show
$$\int_{R^n} e^{-|x|^2} dm = \pi^{n/2}.$$
Hint: For $n = 1$ use
$$(\int e^{-x^2} dx)^2 = \int \int e^{-x^2-y^2} dxdy,$$
and polar coordinates. For $n > 1$, use $|x|^2 = x_1^2 + \ldots + x_n^2$ and Fubini's theorem to reduce to the $n = 1$ case.

4. The convolution $f * g$ of two functions $f, g \in L^1(R)$ is defined as the function
$$f * g(y) = \int_R f(y - x)g(x)dx.$$
Show that $f * g$ is in $L^1(R)$ and
$$\int f * g(y)dy = (\int_R f(x)dx)(\int_R g(x)dx).$$

5. If $f \in L^1(R)$ define its Fourier transform as
$$\hat{f}(t) = \int_R f(x)e^{-ixt} dx.$$
Show that
$$\widehat{f * g}(t) = \hat{f}(t)\hat{g}(t).$$