MAT 319 & 320 Fall 2021, Lecture 5, Thursday, Sept. 9, 2021

Section 2.4: Applications of the Supremum Property

1. Questions from last time?
2. Supremum are compatible with algebraic properties of $\mathbb{R}$

$$\sup(a+S) = a + \sup S$$

3. Suppose A, B are sets and $a \leq b$ for all $a \in A$ and $b \in B$. Then $\sup A \leq \sup B$.
4. If $A \subset B$ then $\sup A \leq \sup B$.
5. $\inf A = -\sup(-A)$ where $-A = \{-a : a \in A\}$.
6. **Definition:** A function $f : D \to \mathbb{R}$ is **bounded above** if $f(D) = \{f(x) : x \in D\}$ is bounded above. Similarly for bounded below and bounded.
7. If $f(x) \leq g(x)$ for all $x \in D$ then

$$\sup_{x \in D} f(x) \leq \sup_{x \in D} g(x).$$

For brevity we will sometimes write

$$\sup_D f \leq \sup_D g.$$

8. This does not imply any relation between $\sup_D f$ and $\inf_D g$.
9. If $f(x) \leq g(y)$ for all $x, y \in D$ then we can conclude

$$\sup_{x \in D} f(x) \leq \inf_{x \in D} g(x).$$

10. **Archimedean property:** If $x \in \mathbb{R}$ then there is a $n \in \mathbb{N}$ such that $x < n$.
 Proven last time.
11. **Corollary 2.4.4:** If $S = \{1/n : n \in \mathbb{N}\}$ then $\inf S = 0$.
 Proof: Clearly 0 is a lower bound for s. If x was a larger lower bound, then $x \leq 1/n$ for all $n \in \mathbb{N}$, so $n \leq 1/x$, so $\mathbb{N}$ would be bounded above.
12. **Corollary 2.4.5:** If $t > 0$ there is an $n \in \mathbb{N}$ so that $1/n < t$.
13. **Corollary 2.4.6:** if $y > 0$ there is an $n \in \mathbb{N}$ so that $n - 1 \leq y < n$.
 Proof: By the Archimedean property, $E = \{n \in \mathbb{N} : y < n\}$ is not empty. By the well ordering property E has a smallest element n. Thus $n - 1 \notin E$. Hence $n - 1 \leq y < n$.
14. **Theorem 2.4.7:** There exists a positive $x \in \mathbb{R}$ such that $x^2 = 2$.
 Proven last time.
15. **The Density Theorem:** If $x < y$ are real numbers, then there is a rational $r \in \mathbb{Q}$ so that $x < r < y$.

17. **Proof:** Without loss of generality (WLOG) we may assume $x > 0$. Since $y > x$, we have $y - x > 0$ so we may choose $n \in \mathbb{N}$ so that $\frac{1}{n} < y - x$. Hence

$$1 < ny - nx$$

$$1 + nx < ny$$

Choose a $m \in \mathbb{N}$ so that $m - 1 \leq 1 + nx \leq m$. Then

$$m - 1 \leq 1 + nx < ny$$

$$m \leq nx < ny$$

Thus

$$nx \leq m \leq ny,$$

$$x \leq \frac{m}{n} \leq y.$$

17. **Corollary 2.4.9:** If $x < y$ are real numbers, then there is an irrational number z with $x < z < y$.

Proof: By Theorem 2.1.4 (Tuesday 8/31) $\sqrt{2}$ is irrational. Using the Density Theorem choose a rational r so that

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}.$$

Then

$$x < r\sqrt{2} < y,$$

and $r\sqrt{2}$ is irrational. (Why?)

Section 2.5: Intervals

18.

open interval $(a, b) = \{x \in \mathbb{R} : a < x < b\}$.
closed interval $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.
half-open (or half-closed) interval $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$.

$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$.
infinite open intervals $(a, \infty) = \{x \in \mathbb{R} : x > a\}$

$(-\infty, a) = \{x \in \mathbb{R} : x < a\}$
infinite closed intervals $[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$

$(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$
Warning: $[a, \infty]$ is not an interval.

19. **Characterization of intervals:** Suppose $S \subset \mathbb{R}$ has the property that $x, y \in S$ and x, y implies $[x, y] \subset S$. Then S is an interval.

Proof: There are four cases depending on whether S is bounded above and below.

Case 1: S is bounded above and below. Let $a = \inf S$, $b = \sup S$. Suppose $a < z < b$. Then there is a $x \in S$ with $x < z$ and a $y \in S$ with $y > z$. Hence $z \in [x, y] \subset S$. Thus $(a, b) \subset S$. Thus S is an interval (one of four possibilities).

Case 2: S is bounded above but not below. Let $b = \sup S$. Suppose $< b$. Since S is not bounded below there is an $x \in S$ with $x < z$. Since $z < b$ we can choose a $y \in S$ with $y > z$. Hence $z \in [x, y] \subset S$. Thus $(-\infty, b) \subset S$. Thus S is an interval (one of two possibilities).

Cases 3 and 4 left as exercises.

20. **Definition:** A sequence of intervals $\{I_n\}_1^\infty$ is called **nested** if

$$I_1 \supset I_2 \supset \dots$$

(usually called **nested decreasing**; nested increasing is $I_1 \subset I_2 \subset \dots$).

21. Bounded nested sequence of intevals can have empty intersection: $\{(0, \frac{1}{n})\}$

22. Unbounded nested sequence of closed intevals can have empty intersection: $\{[n, \infty)\}$.

22. **Nested interval property:** If $I_n = [a_n, b_n]$ is a nested (decreasing) sequence of bounded, closed intervals has non-empty intersection,i.e., there is a a $\xi \in \mathbb{R}$ with $\xi \in I_n$ for all n.

Proof: We have $a_1 \leq a_2 \leq \dots \leq b_1$ is bounded above, so $A = \{a_n : n \in \mathbb{N}\}$ is a bounded set and so has a least upper bound ξ.

By definition, $\xi \leq a_n$ for all n.

We claim that $\xi \leq b_n$ for every n. It suffices so show b_n is an upper bound for A. To see this note that if $k \leq n$, then

$$a_k \leq a_n \leq b_n.$$

On the other hand, if $k \geq n$ then $I_k \subset I_n$, so

$$a_k \leq b_k \leq b_n.$$

Thus b_n is always an upper bound, as desired. $\square$

23. **Theorem 2.5.3:** If $I_n = [a_n, b_n]$ is a nested (decreasing) sequence of bounded, closed intervals whose lengths tend to zero, i.e.,

$$\inf\{b_n - a_n : n \in \mathbb{N}\} = 0$$

then $\xi = \cap_n I_n$ is a single point.

Proof: If η is a distinct point in the intersection, first suppose $\eta > \xi$. Then

$$a_n \leq \xi - \eta < b_n$$

so

$$b_n - a_n > \eta - \xi > 0,$$

so the infimum is not zero. Thus $\eta > \xi$ is impossible. If $\eta < \xi$ the same proof gives

$$b_n - a_n > \xi - \eta > 0.$$

Thus we can't have a second point η in the intersection. $\square$

24. **Theorem 2.5.4:** $\mathbb{R}$ is uncountable.

Proof 1: Suppose $I = [0, 1]$ is countable, say $I = \{x_1, x_2, \dots\}$. Choose a closed bounded interval I_1 so that $x_1 \notin I_1$ (for example, if $x_1 > 0$ take $[0, x_1/2]$ and if $x_1 = 0$ take $[1/2.1]$).

In general, choose a closed bounded interval $I_n \subset I_1$ that does not contain $x_1, \dots, x_n$ (why is this possible?). Then $\cap I_n$ is non-empty so it contains a real number $y \in I$. But $y \in I_n$ implies $y \neq x_n$. This is true for every n so y is not on the list $x_1, x_2, \dots$. This contradicts that I was the whose list.

25. **Proof of claim:** Given a finite set $\{x_1, \dots, x_n\} \subset [0, 1]$ and a closed interval $J \subset [0, 1]$, find a closed interval inside J that does not contain any of the x_k's

Suppose $J = [a, b]$ and choose $2n$ points subintervals $a < y_1 < y_2 < \dots y_{2n} < b$. This defines $2n + 1$ subintervals

$$[a, y_1], [y_1, y_2], \dots, [y_{2n}, b],$$

and each x_k is in at most 2 of these. Thus at least one of these intervals does not contain any of the x_k's.

26. Every $x \in [0, 1]$ has a binary repreentation

$$x = (.a_1 a_2 \dots)$$

defined by recursively bisecting $[0, 1]$ and seeing if $x$is in the left or right half of the parent interval.

27. Binary expansion need not be unique when x is the midpoint of such a bisection, i.e.,

$$.0111111111 \cdots = .100000000000 \dots$$

28. Decimal representation are similar, but we subdivide $[0, 1]$ into ten subintervals labeled $0, 1, \dots 9$. Again, representations need not be unique:

$$.9999999 \cdots = 1.0000000 \dots$$

29. Only countably many reals have non-unique representations. Every non-unique expansion ends in all zeros or all nines. It is determined by the finite segment of digits that come before the repeating 0's or 9's and there are only countbly many finite segments. (Why? use $\mathbb{N}^k$ is countable and countable unions of countable sets are countable).

30. A number is rational iff its decimal representation is periodic (repeats after some point).

31. Decimal representations can be used to give a second proof that $[0, 1]$ is uncountable.

Proof: Assume $[0,1] = x_1, x_2, \ldots$ and write the decimal expansions

$$\begin{aligned}
x_1 &= 0.b_{11}b_{12}b_{13}b_{14}\ldots b_{1n}\ldots \\
x_2 &= 0.b_{21}b_{22}b_{23}b_{24}\ldots b_{2n}\ldots \\
x_3 &= 0.b_{31}b_{32}b_{33}b_{34}\ldots b_{3n}\ldots \\
x_4 &= 0.b_{41}b_{42}b_{43}b_{44}\ldots b_{4n}\ldots \\
x_1 &= 0.b_{51}b_{52}b_{53}b_{54}\ldots b_{5n}\ldots \\
\vdots &= \vdots
\end{aligned}$$

Now define $y = .a_1a_2a_3\ldots$ by choosing a_n so that

$$a_n \neq b_{nn} \quad \text{and} \quad a_n \in \{2,3\}.$$

Then this decimal expansion is not on the list, and y does not have a different expansion that might be on the list, since this one does not end in 0's or 9's.

32. **Cantor's continuum hypothesis:** every infinite subset of $\mathbb{R}$ has a bijection to either $\mathbb{N}$ or $\mathbb{R}$.

Gödel (1930's) and Cohen (1960') showed this is independent of the other axioms of the real numbers. You may assume it is either true or false. Most working analysts accept it as true, but few theorems depend on this assumption. There are various arguments for and against believing the hypothesis, based on whether you prefer statements that follow from the hypothesis or its negation.