MAT 319 & 320 Fall 2021, Lecture 10, Thursday, Sept. 23, 2021

Section 3.5: The Cauchy criterion

1. **Defn:** A real sequence $\{x_n\}$ is a **Cauchy sequence** if for every $\epsilon > 0$ there is a $H(\epsilon) \in \mathbb{N}$ so that for all $n, m \geq (\epsilon)$, we have $|x_n - x_m| < \epsilon$.
2. Definition of convergence says sequence is eventually in any small disk around limit point. Cauchy condition says sequence is eventually in some small disk, not necessarily with same center.
3. Example: $\{1/n\}$
4. Example: $\{(-1)^n\}$
5. **Lemma 3.5.3:** A convergent sequence is Cauchy.

Proof: Suppose $\{x_n\}$ converges to x. Given $\epsilon > 0$ there is an H so that $|x_n - x| < \epsilon/2$ for all $n \geq H$. Thus $n, m \geq H$ implies

$$|x_n - x_m| \leq |x_n - x| + |x_m - x| < \epsilon. \quad \square$$

6. **Lemma 3.5.4:** A Cauchy sequence is bounded.

Proof: Take $\epsilon = 1$. By definition there is an H so that $n.m > H$ implies $|x_n - x_m| < 1$. Hence

$$|x_n| \leq 1 + \max\{|x_1|, \dots, |x_H|\}. \quad \square$$

7. **Cauchy Convergence Criterion:** A real sequence is convergent iff it is Cauchy.

Proof: A convergent sequence is Cauchy by Lemma 3.5.3.

Conversely, suppose $\{x_n\}$ is Cauchy. By Lemma 3.5.4 the sequence is bounded. By the Bolzano-Weierstrass theorem, there is a convergent subsequence $\{x_{n_k}\}$, say with limit x. We claim that $\{x_n\}$ also converges to x.

Since the subsequence converges, for any $\epsilon > 0$, choose H_1 so that $|x_{n_k} - x| < \epsilon/2$ for $n_k \geq H_1$.

Since $\{x_n\}$ is Cauchy we can choose H_2 so that $|x_n - x_m| < \epsilon/2$ for $n, m \geq H_2$.

Let $H = \max(H_1, H_2)$. Then for $n \geq H$ choose some $n_k \geq H$. Then

$$|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence $\{x_n\}$ converges to x.

8. Example: $x_1 = 1, x_2 = 2$, $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$.
9. Example: $\{\sum_{k=1}^{n} \frac{1}{h}\}$ diverges.

Proof: Consider $2^m < k \leq\leq 2^{m+1}$. Then

$$\frac{1}{k} \geq 2^{-m-1}$$

and there are 2^m such terms, so

$$x_{2^{m+1}} - x_{2^m} = \sum_{k=2^m+1}^{2^{m+1}} \frac{1}{k} \geq 2^m \cdot 2^{-m-1} = \frac{1}{2}.$$

So the differences do not tend to zero, so not Cauchy.
10. **Defn:** A real sequence is **contractive** if there is a $0 < C < 1$ so that

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|$$

11. **Theorem 3.5.8:** Every contractive sequence is Cauchy, hence convergent.
Proof: By induction we can show

$$\begin{aligned}
|x_{n+2} - x_{n+1}| &\leq C|x_{n+1} - x_n| \\
&\leq C^2|x_n - x_{n-1}| \\
&\vdots \\
&\leq C^n|x_2 - x_1|.
\end{aligned}$$

Therefore

$$\begin{aligned}
|x_m - x_n| &\leq |x_m - x_{m-1}| + \ldots |x_{n+1} - x_n| \\
&\leq (C^{m-2} + \cdots + C^{n-2})|x_2 - x_1| \\
&\leq C^{n-1}(C^{m-1} + \cdots + 1)|x_2 - x_1| \\
&= C^{n-1}\frac{1 - C^{m-1}}{1 - C}|x_2 - x_1| \\
&\leq C^{n-1}\frac{1}{1 - C}|x_2 - x_1|
\end{aligned}$$

Since $0 < C < 1$, $C^n \to 0$. Therefore this is a Cauchy sequence.
12. **Corollary 3.5.10:** If $\{x_n\}$ is a contractive sequence with constant $0 < C < 1$ and $x = \lim x_n$, then
 (i) $|x - x_n| \leq \frac{C^{n-1}}{1-C}|x_2 - x_1|$.
 (ii) $|x - x_n| \leq \frac{C}{1-C}|x_n - x_{n-1}|$.
27. $f : \mathbb{R} \to \mathbb{R}$ is a C-contraction (or C-Lipschitz) if $|f(x) - f(y)| \leq C|x - y|$.
 Such a map is automatically continuous.
 This happens if f is differentiable and $|f'| \leq C$ by the mean value theorem.
13. **Theorem:** If $C < 1$ and f is a C-contraction, then $f(x) = x$ has a unique solution.
Proof: Take $x_1 = 0$ (any value would work), and define $x_{n+1} = f(x_n)$. Then

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leq C|x_n - x_{n-1}|,$$

so the sequence is contractive, and has a limit x. Then

$$x = \lim x_n = \lim x_{n+1} = \lim f(x_n) = f(x).$$

Hence a solution exists.
 If x, y are two different solutions, then

$$|x - y| = |f(x) - f(y)| \leq C|x - y| < |x - y|.$$

since $0 < C < 1$ and $|x - y| > 0$. Therefore there is at most one solution. $\square$

Section 3.6: Properly divergent sequences Theorem

1. What does $\lim x_n = \infty$ mean?
2. **Defn:**
We say $x_n \to +\infty$ and write $\lim x_n = +\infty$ if for all $a \in \mathbb{R}$ there is a K so that $n \geq K$ implies $x_n > a$.
Similarly for $x_n \to -\infty$: for all $a \in \mathbb{R}$ there is a K so that $n \geq K$ implies $x_n < a$.
We say (x_n) is properly divergent if either of these hold.
3. Examples: $x_n = n$, $x_n = \log n$, $x_n = e^n$
4. **Theorem 3.6.3:** A monotone sequence is properly divergent iff it is unbounded.
Proof: We already know a bounded monotone sequence converges. Conversely, first assume (x_n) is increasing and unbounded. Then for any K there is an n so that $x_n > k$ and since (x_n) is increasing $x_m \geq x_n > K$ for all $m \geq n$. Thus (x_n) diverges property to ∞.
The decreasing case is similar.
5. **Theorem 3.6.4:** Suppose (x_n) and (y_n) are sequences and $x_n \leq y_n$ for all $n \in \mathbb{N}$.
(a) $\lim x_n = +\infty$ then $\lim y_n = +\infty$.
(a) $\lim y_n = -\infty$ then $\lim x_n = -\infty$.
6. **Theorem:** Suppose (x_n) and (y_n) are sequences and suppose

$$\liminf \frac{x_n}{y_n} \geq L > 0.$$

Then $\lim y_n = +\infty$ implies $\lim x_n = +\infty$. If

$$\limsup \frac{x_n}{y_n} \leq L < \infty.$$

Then $\lim x_n = +\infty$ implies $\lim y_n = +\infty$. If
Proof: We do the first statement. Since $L/2 < L$, the definition of $\liminf$ says that for n large enough $(x_n/y_n) > L/22$, or

$$x_n > L \cdot y_n/2.$$

Fix $K > 0$. Since $y_n \to \infty$ we can choose N so that $n \geq N$ implies $y_n \geq 2K/L$. Then $n \geq N$ implies $x_n \geq Ly_n/2 \geq (L(2K/L)/2 = K$. Thus $x_b \to +\infty$.
The other direction is similar.
7. **Theorem 3.6.5:** Suppose (x_n) and (y_n) are sequences and suppose

$$\lim \frac{x_n}{y_n} = L > 0.$$

Then $\lim y_n = +\infty$ iff $\lim x_n = +\infty$.
Proof: This is immediate from previous result.

Section 3.7: Introduction to infinite series

1. **Defn:** given a real sequence (x_n), the corresponding infinite series is the sequence

$$s_n = \sum_{k=1}^{n} z_k = x_1 + \cdots + x_n.$$

These are called the partial sums of the series. The infinite series is denoted

$$\sum_{k=1}^{\infty} x_n.$$

The series converges if the sequence of partial sums converge.
2. **Example:** The geometric series $\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \ldots$.

$$\begin{aligned} s_n &= 1 + r + r^2 + \cdots + r^n \\ r \cdot s_n &= r + r^2 + \cdots + r^n + r^{n+1} \\ (1-r)s_n &= s_n - r \cdot s_n = 1 - r^{n+1} \\ s_n &= \frac{1 - r^{n+1}}{1 - r} \\ \sum_{0}^{\infty} r_n &= \frac{1}{1-r}, \text{ if } -1 < r < 1. \end{aligned}$$

3. **Theorem 3.7.3:** If $\sum_0^\infty x_n$ converges then $|x_n| \to 0$.
11. **Cauchy Criterion:** The series $\sum x_n$ converges iff for every $\epsilon > 0$ there is an $M \in \mathbb{N}$ so that if $m > n \geq M$ then

$$|s_n - s_m| = |x_{n+1} + \cdots + x_m| < \epsilon.$$

4. **Corollary:** If $\sum |x_n|$ converges, then $\sum x_n$ converges.
Idea of proof:

$$|x_{n+1} + \cdots + x_m|||x_{n+1}| + \cdots + |x_m|| < \epsilon.$$

5. **Theorem 3.7.5:** Suppose (x_n) is a non-negative sequence. Then $\sum x_n$ converges iff the partial sums are bounded.
6. $\sum \frac{1}{n}$ diverges.
7. $\sum \frac{1}{n^p}$ converges if $p > 1$.
Enough to show partial sums are bounded.
Enough to show partial sums have bounded subsequence.
Consider s^{2^k}.
Consider

$$s_{2^{k+1}} - s_k = \sum_{n=2^k}^{2^{k+1}} n^{-p} \leq 2^k (2^{-p})^k \leq 2^k (2^{-p})^k = (2^{1-p})^k.$$

so if we let $r = 2^{1-p} < 1$, then

$$s_{2^k} = 1 + r + r^2 + \dots r^k \leq \frac{1}{1-r}. \quad \square$$

8. **Alternating series test:** If (x_n) is positive and decreases to zero, then $\sum(-1)^n x_n$ converges.

Proof: Note that if n is odd then $(-1)^n = -1$ so

$$s_n = s_{n-1} + (-1)^n x_n = s_{n-1} - x_n < s_{n-1}.$$

Also

$$s_n = s_{n-2} + x_{n-1} - x_n > s_{n-2}.$$

Hence

$$s_{n-2} < s_n < s_{n-1}.$$

Similarly if n is even, then

$$s_{n-1} < s_n < s_{n-2}.$$

So the intervals I_n with endpoints s_{n-1}, s_n are nested, bounded have diameters tending to zero, and I_n contain all the partials sums s_m with $m \geq n$. Thus the partial sums are a Cauchy sequence, hence converges. $\square$

9. **Theorem 3.7.7:** Suppose (x_n) and (y_n) are real sequences and that for some K, $0 \leq x_n \leq y_n$ for $n \geq K$. Then
   (a) if $\sum y_n$ converges, so does $\sum x_n$.
   (b) if $\sum x_n$ diverges, so does $\sum y_n$.

10. **Limit comparison test:** Suppose (x_n) and (y_n) are strictly positive sequences and

$$r = \lim \frac{x_n}{y_n},$$

exists. Then
   (a) if $r \neq 0$ then $\sum x_n$ converges iff $\sum y_n$ does.
   (b) if $r = 0$ and $\sum y_n$ converges then $\sum x_n$ converges.

20. Suppose $f : \mathbb{N} \to \mathbb{N}$ is a bijection. The

$$\sum_{n=1}^{\infty} x_{f(n)}$$

is called a re-arrangement of $\sum x_n$.

11. **Theorem:** If x_n are non-negative and $\sum |x_n|$ converges, any re-arrangement then $\sum x_n$ converges to same limit.

False in general: $\sum x_n$ converges but $\sum |x_n|$ diverges, then $\sum x_n$ can be re-arranged to sum to any limit we want.