MAT 319 & 320 Fall 2021, Lecture 9, Tuesday, Sept. 21, 2021

Section 3.4: Subsequences and the Bolzano-Weierstrass Theorem

1. Questions from last time?
2. **Defn of subsequence:** If $\{x_n\}$ is a sequence, and $n_1 < n_2 < \dots$ then $x_{n_1}, x_{n_2}, \dots$ is a subsequence.
3. Subset is not the same as subsequence.
4. $n_k \geq k$. Proof by induction.
5. **Theorem 3.4.2:** If $\{x_n\}$ converges to x, then any subsequence also converges to x.

Proof: Suppose $\epsilon > 0$ is given. Since $\{x_n\}$ converges to x there is a $K(\epsilon)$ so that $k > K(\epsilon)$ implies $|x_k - x| < \epsilon$. Since $n_k \geq k$ we also have $|x_{n_k} - x| < \epsilon$ for $k > K(\epsilon)$. Hence $x_{n_k} \to x$.

6. **Theorem 3.4.4:** Let $\{x_n\}$ be a sequence of real numbers. TFAE
   (i) The sequence does not converge to x.
   (ii) There is $\epsilon_0 > 0$ so that for any $k \in \mathbb{N}$ there is $n_k > k$ so that $|x_{n_k} - x| \geq \epsilon_0$.
   (iii) There is $\epsilon_0 > 0$ and a subsequence $\{x_{n_k}\}$ so that $|x_{n_k} - x| \geq \epsilon_0$ for all k..

Proof:

(i) $\Rightarrow$ (ii): This the contrapositive of the definition of convergence to x.

(i) $\Rightarrow$ (iii): Let ϵ_0 be as in (ii) and choose n_1 so that $|x_{n_1} - x| \geq \epsilon_0$. Then choose $n_2 > n_1$ be such that $|x_{n_2} - x| \geq \epsilon_0$. In general choose $n_{k+1} > n_k$ so that $|x_{n_{k+1}} - x| \geq \epsilon_0$. This gives the desired subsequence.

(iii) $\Rightarrow$ (i): This follows from Theorem 3.4.2: if $\{x_n\}$ converges to x so does any subsequence. So if the subsequence does not converge, then $\{x_n\}$ does not either.

7. **Divergence Criteria:** If $\{x_n\}$ has either of these properties then it diverges:
   (i) there are two subsequence that converge to different limits.
   (ii) the sequence is unbounded.
8. Example: $(-1)^n$
9. Example: $\sin(n)$
10. **Monotone Subsequence Theorem:** Every real sequence contains a monotone subsequence.

Proof: Given $\{x_n\}$ say x_m is a "peak" if $x_n \leq x_m$ for all $n > m$.

Case 1: $\{x_n\}$ has infinitely many peaks. List them: $x_{m_1}, x_{m_2}, \dots$. This is decreasing subsequence.

Case 2: $\{x_n\}$ has finitely many peaks. Let x_{n_1} be past the last peak. This point is not a peak so there is a $n_2 > n_1$ so that $x_{n_2} > x_{n_1}$. But x_{n_2} is not a peak either so there is a $n_3 > n_2$ so that $x_{n_3} > x_{n_2}$. Continuing in this inductively gives an increasing sequence.

11. **The Bolzano-Weierstrass Theorem:** Every bounded sequence has a convergent subsequence.
bf Proof: By the previous result there is a monotone subsequence. By Thm 3.3.2 (Monotone convergence theorem) this subsequence converges. $\square$
A second proof based on nested interval is also given in the text. Based on bisecting and taking subinterval containing infinity many elements.
12. **Theorem 3.4.9:** A bounded sequence of real numbers converges iff every convergent subsequence has the same limit.

See proof in text.

13. Suppose $\{x_n\} \subset \mathbb{R}$ is bounded

The "limit superior" (or limsup) is the infimum of $v \in \mathbb{R}$ so that $x_n \leq v$ except for finitely many n (so the sequence is eventually $\leq v$.

The "limit inferior" (or liminf) is the supremum of $w \in \mathbb{R}$ so that $x_n \geq w$ except for finitely many n (so the sequence is eventually $\geq w$.

14. **Theorem 3.4.11:** If $\{x_n\}$ is bounded then TFAE:

(a) $x = \limsup(x_n)$.

(b) If $\epsilon > 0$ there are finitely many n so that $x_n > x + \epsilon$ and infinitely many so that $x_n > x - \epsilon$.

(c) $x = \inf\{u_m\}$ where $u_m = \sup\{x_n : n \geq m\}$.

(d) If S is the set of all limits of subsequences of $\{x_n\}$, then $x = \sup S$.

Proof:

(a) $\Rightarrow$ (b): Let V be the set of reals v so that $x_n < v$ for all but finitely many n. Since x is the infimum of this set, there is an element of V in $[x, x + \epsilon)$ for any $\epsilon > 0$. So there are only finitely many $x_n > v$ hence only finitely many bigger than $x + \epsilon$. On the other hand $x - \epsilon \notin V$ so there are infinitely many x_n greater than $x - \epsilon$.

(b) $\Rightarrow$ (c): If (b) holds then $u_m < x + \epsilon$ holds for large enough m. Thus $\inf u_m \leq x + \epsilon$ and hence $\inf u_m \leq x$. Since infinitely many $x_n > x - \epsilon$, $u_m \geq x - \epsilon$ for all m. Hence $\inf u_m \geq x - \epsilon$ for all $\epsilon > 0$. Hence $\inf u_m \geq x$. Thus $\inf u_m = x$. .

(c) $\Rightarrow$ (d): Suppose $X' = \{x_{n_k}\}$ is a convergent subsequence. Its limit must be less than u_m for all m, and hence $\lim x_{n_k} \leq \lim u_m = x$. Thus $\sup S \leq x$.

Conversely there is an n_1 so that $u_1 - 1 < x_{n_1} \leq u_1$. Inductively choose $n_{k+1} > n_k$ so that $u_k - \frac{1}{1+k} < x_{n_1} \leq u_k$. Then

$$\lim x_{m_k} = \lim u_k = x,$$

so $x \in S$. Hence $\sup S \geq x$.

(d) $\Rightarrow$ (c): Let $w = \sup S$ and let $\epsilon > 0$. Then $w + \epsilon$ is strictly larger than the limit of any subsequence. So there can only be finitely many $x_n > w + \epsilon$ (otherwise by Bolzano-Weierstrass there would be a subsequence with limit $\geq w + \epsilon$). Thus $w \in V$, so $x \leq w$.

On the other hand, there is a subsequence converging to some number $> w - \epsilon$ so there are infinitely many elements of $x_n > w - \epsilon$. Thus no number less than x in is V. Thus $x = \inf V = \limsup x_n$. $\square$

15. **Theorem 3.4.12:** A bounded sequence is convergent iff $\limsup x_n = \liminf x_n$.

Section 3.5: The Cauchy criterion

16. **Defn:** A real sequence $\{x_n\}$ is a **Cauchy sequence** if for every $\epsilon > 0$ there is a $H(\epsilon) \in \mathbb{N}$ so that for all $n, m \geq (\epsilon)$, we have $|x_n - x_m| < \epsilon$.
17. Definition of convergence says sequence is eventually in any small disk around limit point. Cauchy condition says sequence is eventually in some small disk, not necessarily with same center.
18. Example: $\{1/n\}$
19. Example: $\{(-1)^n\}$
20. **Lemma 3.5.3:** A convergent sequence is Cauchy.
Proof: Suppose $\{x_n\}$ converges to x. Given $\epsilon > 0$ there is an H so that $|x_n - x| < \epsilon/2$ for all $n \geq H$. Thus $n, m \geq H$ implies

$$|x_n - x_m| \leq |x_n - x| + |x_m - x| < \epsilon. \quad \square$$

21. **Lemma 3.5.4:** A Cauchy sequence is bounded.
Proof: Take $\epsilon = 1$. By definition there is an H so that $n.m > H$ implies $|x_n - x_m| < 1$. Hence

$$|x_n| \leq 1 + \max\{|x_1|, \ldots, |x_H|\}. \quad \square$$

22. **Cauchy Convergence Criterion:** A real sequence is convergent iff it is Cauchy.
Proof: A convergent sequence is Cauchy by Lemma 3.5.3.

Conversely, suppose $\{x_n\}$ is Cauchy. By Lemma 3.5.4 the sequence is bounded. By the Bolzano-Weierstrass theorem, there is a convergent subsequence $\{x_{n_k}\}$, say with limit x. We claim that $\{x_n\}$ also converges to x.

Since the subsequence converges, for any $\epsilon > 0$, choose H_1 so that $|x_{n_k} - x| < \epsilon/2$ for $n_k \geq H_1$.

Since $\{x_n\}$ is Cauchy we can choose H_2 so that $|x_n - x_m| < \epsilon/2$ for $n, m \geq H_2$.

Let $H = \max(H_1, H_2)$. Then for $n \geq H$ choose some $n_k \geq H$. Then

$$|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence $\{x_n\}$ converges to x.
23. Example: $x_1 = 1, x_2 = 2$, $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$.
24. Example: $\{\sum_{k=1}^{n} \frac{1}{h}\}$ diverges.
Proof: Consider $2^m < k \leq\leq 2^{m+1}$. Then

$$\frac{1}{k} \geq 2^{-m-1}$$

and there are 2^m such terms, so

$$x_{2^{m+1}} - x_{2^m} = \sum_{k=2^m+1}^{2^{m+1}} \frac{1}{k} \geq 2^m \cdot 2^{-m-1} = \frac{1}{2}.$$

So the differences do not tend to zero, so not Cauchy.
25. **Defn:** A real sequence is **contractive** if there is a $0 < C < 1$ so that

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|$$

25. **Theorem 3.5.8:** Every contractive sequence is Cauchy, hence convergent.
Proof: By induction we can show

$$\begin{aligned} |x_{n+2} - x_{n+1}| &\leq C|x_{n+1} - x_n| \\ &\leq C^2|x_n - x_{n-1}| \\ &\vdots \\ &\leq C^n|x_2 - x_1|. \end{aligned}$$

Therefore

$$\begin{aligned} |x_m - x_n| &\leq |x_m - x_{m-1}| + \dots |x_{n+1} - x_n| \\ &\leq (C^{m-2} + \dots + C^{n-2})|x_2 - x_1| \\ &\leq C^{n-1}(C^{m-1} + \dots + 1)|x_2 - x_1| \\ &= C^{n-1}\frac{1 - C^{m-1}}{1 - C}|x_2 - x_1| \\ &\leq C^{n-1}\frac{1}{1 - C}|x_2 - x_1| \end{aligned}$$

Since $0 < C < 1$, $C^n \to 0$. Therefore this is a Cauchy sequence.
26. **Corollary 3.5.10:** If $\{x_n\}$ is a contractive sequence with constant $0 < C < 1$ and $x = \lim x_n$, then
(i) $|x - x_n| \leq \frac{C^{n-1}}{1-C}|x_2 - x_1|$.
(ii) $|x - x_n| \leq \frac{C}{1-C}|x_n - x_{n-1}|$.
27. $f : \mathbb{R} \to \mathbb{R}$ is a C-contraction (or C-Lipschitz) if $|f(x) - f(y)| \leq C|x - y|$.
Such a map is automatically continuous.
This happens if f is differentiable and $|f'| \leq C$ by the mean value theorem.
28. **Theorem:** If $C < 1$ and f is a C-contraction, then $f(x) = x$ has a unique solution.
Proof: Take $x_1 = 0$ (any value would work), and define $x_{n+1} = f(x_n)$. Then

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leq C|x_n - x_{n-1}|,$$

so the sequence is contractive, and has a limit x. Then

$$x = \lim x_n = \lim x_{n+1} = \lim f(x_n) = f(x).$$

Hence a solution exists.
If x, y are two different solutions, then

$$|x - y| = |f(x) - f(y)| \leq C|x - y| < |x - y|.$$

since $0 < C < 1$ and $|x - y| > 0$. Therefore there is at most one solution. $\square$