**MAT 319 & 320 Fall 2021, Lecture 14, Thursday, Oct 7, 2021**

**First meeting of MAT 320. In P-131 of Math Tower.**

## Section 4.2: Limit Theorems

1. **Defn:** Let $f : A \to \mathbb{R}$ and let $c$ be a cluster point of $A$. We say $f$ is bounded on a neighborhood of $c$ if there is a $\delta > 0$ and $M < \infty$ so that $|f(x)| < M$ for $x \in V_\delta(c) \cap A$.
2. **Theorem 4.2.3:** If $f : A \to \mathbb{R}$ has a limit at $c$ then $f$ is bounded on some neighborhood of $c$.
**Proof:** Suppose $L$ is the limit and take $\epsilon = 1$. Choose $\delta > 0$ as in the definition of limit. Then $|f(x)| \leq |L| + 1$ for $0 < |x - c| < \delta$. So if $c \notin A$ we can take $M = |L| + 1$. If $c \in A$, take $M = \max(|L| + 1, M)$. $\square$
3. **Defn:** If $f, g$ are real valued functions on $A$ we define the function $f + g$ by

$$(f_g)(x) = f(x) + g(x).$$

Similarly for $fg$ and $f/g$ (where $g \neq 0$).
4. **Theorem 4.2.4:** If $f, g : A \to \mathbb{R}$ have a limits $L, M$ at $c$ and $b \in \mathbb{R}$, then
(a)

$$\lim_{x\to c}(f + g) = L + M,$$
$$\lim_{x\to c}(f - g) = L - M,$$
$$\lim_{x\to c}(fg) = LM,$$
$$\lim_{x\to c}(bf) = bL.$$

(b) If $g(x) \neq 0$ for $x \in A$ and $\lim_{x\to c} g = H \neq 0$ then

$$\lim_{x\to c}(f/g) = L/H.$$

**Proof:** This follows from Theorem 3.2.3 (about sums and products of limits of sequences) and Theorem 4.1.8 (about $f$ having a limit at $c$ iff if has a limit along all sequences converging to $c$).
5. **Cor:** If $\lim_{x\to c} f(x) = L$ then $\lim_{x\to c} f^n(x) = L^n$ then
6. If $p$ is a polynomial then $\lim_{x\to c} p(x) = p(c)$
7. If $p, q$ are polynomials and $q(c) \neq 0$ then

$$\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$$

8. **Theorem 4.2.6:** Suppose $f : A \to \mathbb{R}$ and $c$ is a cluster point of $A$. If $a \leq f(x) \leq b$ for all $x \in A \setminus \{c\}$, and $\lim_{x\to c} f(x) = L$ exists, then $a \leq L \leq b$.

**Proof:** If the limit exists then there is a sequence $x_n toc$ with $x_n \neq c$ and $L = \lim x_n$. Thus $a \leq L \leq b$ by Theorem 3.2.6. $\square$

9. **Squeeze Theorem:** Suppose $f, g, h : A \to \mathbb{R}$ and $c$ is a cluster point of $A$. If $f \leq g \leq h$ on $A \setminus \{c\}$ and

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L,$$

then

$$\lim_{x \to c} g(x) = L.$$

**Proof:** follows from Theorem 3.2.7 for sequences. $\square$

10. $\lim_{x \to 0} x^{3/2} = 0$ for $A = \{x > 0\}$.

**Proof:** Note $0 \leq x^{1/2} \leq 1$ for $0 \leq x \leq 1$. Thus

$$0 \leq x^{3/2} \leq x,$$

so limit is 0 by the squeeze theorem.

11. $\lim_{x \to 0} \sin(x) = 0$ for $A = \mathbb{R}$.

**Proof:** We have not even defined $\sin(x)$ yet. However later we will define it and show that

$$|\sin(x)| = |\int_0^x \cos(x) dx| \leq |\int_0^x 1 dx| = |x|,$$

so

$$-x \leq \sin(x) \leq x,$$

so limit is zero by the squeeze theorem.

12. $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ for $A = \mathbb{R}$.

**Proof:** Later we will prove

$$1 - \frac{1}{2}x^2 \leq \cos x \leq 1.$$

Integrating implies

$$x - \frac{1}{6}x^3 \leq \sin x \leq x,$$

so

$$1 - \frac{1}{6}x^2 \leq \frac{\sin x}{x} \leq 1,$$

so

$$\sin(x) \to 0 \text{ as } x \to 0.$$

13. $\lim_{x \to 0} x \sin(\frac{1}{x}) = 0$.

**Proof:** We have

$$-1 \leq \sin(\frac{1}{x}) \leq 1$$

so

$$-|x| \leq x \sin(\frac{1}{x}) \leq |x|$$

so limit is zero by the squeeze theorem.

14. **Theorem 4.2.9:** Suppose $f : A \to \mathbb{R}$ and $c$ is a cluster point of $A$. If $\lim_{x\to c} f = L > 0$ then there is neighborhood $V_\delta(c)$ of $c$ so that $f(x) > 0$ on $A \cap V_\delta(c) \setminus \{c\}$.
**Proof:** Take $\epsilon = L/2 > 0$ and choose $\delta > 0$ so that $0 < |x - c| < \delta$, $x \in A$ implies $|f(x) - L| < \epsilon$. Then

$$f(x) \geq L - \epsilon = L - L/2 = L/2 > 0. \quad \square$$

15. Similar statement for $\lim_{x\to c} f(x) = L < 0$. Deduce by taking $g = -f$.

## Section 4.2: Some extensions of the limit concept

1. One sided limits.
2. Infinite limits at finite points.
3. Limits as $x \to \pm\infty$.

## Section 5.1: Some extensions of the limit concept

1. **Defn:** Let $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ and $c \in A$. We say $f$ is **continuous** at $c$ if for any $\epsilon > 0$ there is a $\delta > 0$ so that $x \in A$, $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$.
   For cluster points of $A$, this is the same as saying $\lim_{x\to c} f(x) = f(c)$.
   For isolated points, $f$ is always continuous.
2. **Theorem 5.1.2:** $f$ is continuous at $c$ iff given any $\epsilon > 0$ there is a $\delta > 0$ so that

$$f(V_\delta(c)) \subset V_\epsilon(f(c)).$$

**Proof:** obvious from definitions.
3. **Sequential criterion for continuity:** $f$ is continuous at $c \in A$ iff for every sequence $x_n \to c$ we have $f(x_n) \to f(c)$.
3. **Discontinuity criterion:** $f$ is discontinuous at $c \in A$ iff there is a sequence $x_n \to c$ so that $f(x_n) \not\to f(c)$.
4. $\sin(1/x)$ is discontinuous at 0. $x_n = 1/\pi n$, $y_n = -1/\pi n$.
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5. We say $f$ is continuous on a set $B \subset A$ if $f$ is continuous at every point of $B$.
6. Examples:
   constant functions, identity function, powers, polynomials
   rational functions, except at poles
   Dirichlet function, discontinuous everywhere
   Thomae's function, discontinuous exactly at rationals
7. If $c$ is a cluster point of $A$ but not in $A$, then a function $f$ on $A$ can be defined at $c$ to be continuous there iff $f$ has a limit as $x \to c$ in $A$.