MAT 319 & 320 Fall 2021, Lecture 13, Tuesday, Oct 5, 2021

Last joint lecture. MAT 320 will meet in P-131 of Math Tower starting on Thursday. MAT 319 will continue in this room with Prof. Martens.

Section 4.1: Limits of functions

1. Discuss cut-off for MAT 320, exam results.
2. Newton, Leibniz 1680's; Cauchy 1821; Weierstrass late 1800's.
3. **Defn:** Let $A \subset \mathbb{R}$. A point $c \in \mathbb{R}$ is a **cluster point** of A if for all $\delta > 0$ there is a $x \in A$ with $x \neq c$ and $|c - x| < \delta$.
4. Finite sets have no cluster points.
5. Set of cluster points of $(0, 1)$ is $[0, 1]$.
6. Set of cluster points of $\mathbb{Q}$ is $\mathbb{R}$.
7. $\mathbb{N}$ has no cluster points.
8. **Theorem 4.1.2:** c is a cluster point of $A \subset \mathbb{R}$ iff there is a sequence in $A \setminus \{c\}$ converging to c.

Proof: If c is a cluster point of A choose x_n in $(c - \frac{1}{n}, c + \frac{1}{n}) \setminus \{c\}$. Then $x_n \to c$ and $x_n \neq c$.

Conversely, if $x_n \to c$ and $x_n \neq c$ for all n, then $x_n \in (c - \frac{1}{n}, v + \frac{1}{n})$ for all n a large enough.

9. **Defn:** Suppose $A \subset \mathbb{R}$, c is a cluster point of A, and $f : A \to \mathbb{R}$. We say $L \in \mathbb{R}$ is a limit of f at c if given any $\epsilon > 0$ there is a $\delta > 0$ so that $0 < |x - c| < \delta$ implies $|f(x) - L| < \epsilon$. We write

$$\lim_{x \to c} f(x) = L.$$

$$f(x) \to L \text{ as } x \to c$$

10. We sometimes write $\delta(\epsilon)$ to emphasize that δ depends on ϵ.
11. **Theorem 4.1.5:** f can have at most one limit at a cluster point c.

Proof: Suppose L_1, L_2 are two different limits of f at c, and let $\epsilon = |L_1 - L_1| > 0$. Choose $\delta_1 = \delta(\epsilon)$ so that $|f(x) - L_1| < \epsilon/2$ if $0 < |x - c| < \delta_1$. Similarly for δ_2 and L_2.

Let $\delta = \min(\delta_1, \delta_2)$ and choose x with $0 < |x - c| < \delta$. Then

$$\epsilon = |L_1 - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \epsilon/2 + \epsilon/2 = \epsilon,$$

which is a contradiction. $\square$

12. Draw picture of limit existing and not existing.
13. **Theorem 4.1.6:** Let $f : A \to \mathbb{R}$ and let c be a cluster point of c. TFAE

(i) $\lim_{x \to c} f(x) = L$.

(ii) Given any ϵ-neighborhood $V_\delta(L)$ of L there is a δ-neighborhood of $V_\delta(c)$ of c so that $f(V_\delta(c) \cap A \setminus \{c\}) \subset V_\epsilon(L)$.

Proof: left to reader. Follow the definitions.
14. Limit of constant function is the same constant.
15. If $f(x) = x$ then $\lim_{x\to c} f(x) = c$.
16. If $f(x) = x^2$ then $\lim_{x\to c} f(x) = c^2$.
17. If $f(x) = 1/x$ and $c \neq 0$ then $\lim_{x\to c} f(x) = 1/c$.
18. If $f(x) = (x^2 - 4)/(x^2 + 1)$ then $\lim_{x\to 2} f(x) = 0$.
19. **Theorem 4.1.8:** (Sequential criterion) Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. TFAE:
(i) $\lim_{x\to c} f(x) = L$.
(ii) For every sequence (x_n) in $A \setminus \{c\}$ converging to c, the sequence $(f(x_n))$ converges to L.
Proof:
(i) $\Rightarrow$ (ii): Assume (i) holds and that (x_n) is a sequence as in (ii). Let $\epsilon > 0$. There is a $\delta > 0$ so that $o0 < |x - c| < \delta$ implies $|f(x) - L| < \epsilon$. There is a K so that $n > K$ implies $|x_n - c| < \delta$, hence $|f(x_n) - L| < \epsilon$. This $f(x_n) \to L$.
(ii) $\Rightarrow$ (i) : Suppose (i) fails. Then there is an $\epsilon > 0$ so that for any δ we can find x with $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon$. So for each n we can set $\delta = 1/n$ and choose x_n so that $0 < |x_n - c| < \delta$ and $|f(x_n) - L| \geq \epsilon$. Clearly $f(x_n) \not\to L$. □
20. **Divergence Criteria:** Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. TFAE
(a) f does not have a limit L at c iff there is a sequence (x_n) in $A \setminus \{c\}$ so that $x_n \to c$ but $f(x_n) \not\to L$.
(b) f does not have any limit at c iff there is a sequence (x_n) in $A \setminus \{c\}$ so that $x_n \to c$ but $f(x_n)$ does not converge.
21. $f(x) = 1/x$ does not have a limit at 0.
22. $f(x) = \mathrm{sign(x) = x/|x|, x \neq 0}$ does not have a limit at 0.
23. $f(x) = \sin(1/x), \neq 0$ does not have a limit at 0.
24. Let $f(x) = 1$ if $x \in \mathbb{Q}$, and $f(x) = 0$ otherwise. Then f does not have a limit at any point.
25. Let $f(x) = 1/q$ if $x = p/q \in \mathbb{Q}$ in lowest terms, and $f(x) = 0$ otherwise. Then $\lim_{x\to c} = 0$ if c is irrational and the limit does not exist if c is rational.