MAT 320 Fall 2021, Thursday, Oct 21, 2021

Finish from Tuesday — Section 5.6: Monotone and Inverse Functions

8. **Continuous Inverse Theorem:** If f is strictly monotone and continuous on an interval I, then f has an inverse g that is strictly monotone and continuous.
Proof: Enough to consider f increasing.

By Theorem 5.3.10, $J = f(I)$ is an interval. Since strictly monotone implies injective, f has an inverse $g : J \to I$.

Easy to see that g is strictly increasing: if $x, y \in J$ and $x < y$ Then $f(g(x)) < f(g(y))$ so $g(x) < g(y)$ since f is increasing.

If g were discontinuous, then it must have a jump discontinuity at some point $c \in J$. This means there is a value between $\lim_{x\to c^-} g$ and $\lim_{x\to c^+} g$ which is not in the image of g. Thus $g(J) = I$ is not an interval, a contradiction. $\square$

9. nth roots exist for $x \geq 0$ if n is even.
10. nth roots exist for $x \in \mathbb{R}$ if n is even.
11. positive rational powers x^r exist for $x \geq 0$.
12. all rational powers x^r exist for $x > 0$.

Section 6.1: The derivative

1. **Defn:** Suppose f is defined on an interval I and $c \in I$. We say L is the derivative of f at c if

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = L.$$

We allow c to be an endpoint of I.

We say f is differentiable at c and write $f'(c) = L$.

2. If f is differentiable on all of I we obtain a function f' on I.
3. **Theorem 6.1.2:** If f has a derivative at c, then f is continuous at c.

Proof: Write

$$f(x) - f(c) = (x - c)\frac{f(x) - f(c)}{x - c}$$

$$\lim_{x\to c} f(x) - f(c) = \left(\lim_{x\to c}(x - c)\right) \cdot \left(\lim_{x\to c} \frac{f(x) - f(c)}{x - c}\right) = 0 \cdot f'(c) = 0.$$

Thus f is continuous at c. $\square$

4. There are continuous functions that are nowhere continuous, first constructed by Weierstrass, e.g.,

$$\sum_{n=0}^{\infty} 2^{-n} \cos(3^n).$$

See Chapter 5 of my book *Fractals in probability and analysis*, PDF on my webpage.

5. **Theorem 6.4.1:** Suppose f, g are defined on an interval I and differentiable at $c \in I$. Then
   (a) if $a \in \mathbb{R}$ then af is differentiable and $(af)'(c) = af'(c)$.
   (b) $f + g$ is differentiable at c and $(f+g)'(c) = f'(c) + g'(c)$.
   (c) fg is differentiable at c and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.
   (d) f/g is differentiable at c and $(f/g)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$.

Proof: See text (same proof as in calculus classes).

6. Deduce power rule, rule for finite products.

7. **Carathéodory's Theorem:** Suppose f is defined on an interval I and $c \in I$. Then f is differentiable at c iff there is a function φ on I that is continuous at c and satisfies

$$f(x) - f(c) = \varphi(x)(x - c).$$

Proof:

If f is differentiable at c set

$$\varphi(x) = \frac{f(x) - f(c)}{x - c}$$

for $x \neq c$ and $\varphi(c) = f'(c)$. It is easy to check φ has desired properties.

Conversely, if such a φ exists, then

$$\varphi(c) = \lim_{x \to c} \varphi(x) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

so $f'(c)$ exists and equals $\varphi(c)$. $\square$

8. **The chain rule:** Let I, J be intervals in $\mathbb{R}$ and suppose $f : J \to I$ and $g : I \to \mathbb{R}$. If f is differentiable at $c \in I$ and g is differentiable at $g(c)$ then the composition $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = (g'(f(c)) \cdot f'(c).$$

Proof: By Carethéodory's theorem there is function φ continuous at c so that $\varphi(c) = f'(c)$ and

$$f(x) - f(c) = \varphi(x)(x - c).$$

Similarly, there is a function ψ continuous at $f(c)$ so that $\psi(c) = g'(f(c))$ and

$$g(y) - g(f(c)) = \psi(y)(y - c).$$

Then

$$g(f(x)) - g(f(c)) = \psi(f(c))(f(x) - f(c)) = (\psi \circ f)(x) \cdot \varphi(x)(x - c).$$

Now apply Carathéodory's theorem to $g \circ f$ with the function $\Phi = (\psi \circ f) \cdot \varphi$ to prove $g \circ f$ is differentiable with derivative $(\psi \circ f)(c) \cdot \varphi(c) = g'(f(c))f'(c)$. $\square$

9. **Examples:**

Power rule for integers.

Derivative of $1/f$. Quotient rule.

$|x|$

10. **Theorem 6.1.8:** Suppose f is strictly monotone and continuous on an interval I. Let $J = f(I)$ and let $g : J \to I$ be the inverse to f. If f is differentiable at c and $f'(c) \neq 0$, then g is differentiable at $d = f(c)$ and

$$g'(d) = \frac{1}{f'(c)} = \frac{1}{f'(g(d))}.$$

Proof: By Carathéodory's theorem

$$f(x) - f(c) = \varphi(x)(x - c)$$

and $\varphi(c) = f'(c) \neq 0$. Thus φ is non-zero on some δ-neighborhood $V = (c - \delta, c + \delta)$ (thm 4.2.9).

Let $U = f(V \cap I)$. Then $f(g(y)) = y$ on U so

$$y - d = f(g(y)) - f(c) = \varphi(g(y)) \cdot (g(y) - g(d)).$$

Since φ is non-zero, we can divide by it to get

$$\frac{y - d}{\varphi(g(y))} = g(y) - g(d).$$

Now apply Carathéodory's theorem using $1/\varphi(g(y))$, which is continuous at d to deduce g is differentiable with derivative $1/\varphi(g(d)) = 1/f'(c)$. $\square$

11. **Theorem 6.1.9:** Suppose f is strictly monotone and continuous on an interval I. Let $J = f(I)$ and let $g : J \to I$ be the inverse to f. If f is differentiable on I and f', is never zero, then g is differentiable on I and $g' = 1/f'(g(x))$.

12. **Examples:**

$x^{1/n}$

$x^{p/q}$

$\arcsin(x)$

Section 6.2: The Mean Value Theorem

1. **Defn:** f has a relative (or local) maximum at $c \in I$ if $f(x) \leq f(c)$ for all x in some δ-neighborhood of c.

Similar for minimum. Extremum = minimum or maximum.

2. **Interior Extremum Theorem:** Suppose $c \in I$ is an interior point of I at which $f : I \to \mathbb{R}$ has a relative extremum. If f is differentiable at c then $f'(c) = 0$.

Proof: We only consider case when f has a local maximum at c.

For x close enough to c, $f(x) - f(c) \leq 0$, so

$$\frac{f(x) - f(c)}{x - c} \leq 0 \text{ if } x > c$$

$$\frac{f(x) - f(c)}{x - c} \geq 0 \text{ if } x < c.$$

So if limit exists, it must be zero. $\square$

3. **Corollary 6.2.2:** Suppose $c \in I$ is an interior point of I at which $f : I \to \mathbb{R}$ has a relative extremum. Then either $f'(c) = 0$ or f is not differentiable at c.
3. **Rolle's Theorem:** Suppose f is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b) = 0$. Then there is at least one point $c \in (a, b)$ where $f'(c) = 0$.
Proof: If f is always zero, any c will work.

Otherwise, by replacing f by $-f$ if necessary, we may assume f takes some positive value. By Theorem 5.3.4 f attains a positive maximum value at an interior point c and hence $f'(c) = 0$ by the previous result. □

4. **Mean Value Theorem:** Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there is at least one point $c \in (a, b)$ where

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof: Apply Rolle's theorem to

$$\varphi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Check that $\varphi(a) = \varphi(b) = 0$ and φ is differentiable where f is, so $\varphi'(c) = 0$ for some c. Hence

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a}$$

or

$$f'(c)(b - a) = f(b) - f(a). \quad \square$$

5. **Theorem 6.2.5:** Suppose f is continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = 0$ for all $x \in (a, b)$ then f is constant.
Proof: If there is $x \in (a, b)$ where $f(x) \neq f(a)$ then by the Mean Value Theorem there is a $c \in (a, x)$ where

$$f'(c) = \frac{f(x) - f(a)}{x - a} \neq 0,$$

a contradiction. □

6. **Corollary 6.2.6:** Suppose f, g are continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = g'(x)$ for all $x \in (a, b)$ then $f = g + C$ where $C = f(a) - g(a)$ is constant.
Proof: $(f - g)' = 0$ so $f - g$ is constant and equal to $f(a) - g(a)$ at a. □

7. **Theorem 6.2.7:** Suppose $f : I \to \mathbb{R}$ is differentiable on I. Then
(a) f is increasing iff $f'(x) \geq 0$ for all $x \in I$.
(b) f is decreasing iff $f'(x) \leq 0$ for all $x \in I$.

Proof:
(a) Suppose $f' \geq 0$. If $x < y$ are in I then

$$f(y) - f(x) = f'(c)(y - x) \geq 0,$$

so f is increasing.

Conversely, if f is increasing then $x \neq y$ implies (two cases)

$$\frac{f(y) - f(x)}{y - x} \geq 0,$$

so the limit as $x \to y$ is ≥ 0. Thus $f'(y) \geq 0$ for any $y \in I$.

Part (b) is proved similarly (or apply Part (a) to $g = -f$).

8. A differentiable f is strictly increasing if $f' > 0$. Converse is not true because of x^3.

9. $f'(x) > 0$ does not imply f is increasing on any neighborhood of x.

10. **First Derivative Test for Extrema:** Suppose f is continuous on $I = [a, b]$, that $c \in I$ is an interior point and f is differentiable on (a, c) and (c, b). Then

(a) If there is a $\delta > 0$ so that $f' \geq 0$ on $(c - \delta, c)$ and $f' \leq 0$ on $(c, c + \delta)$ then f has a relative maximum at c.

(b) If there is a $\delta > 0$ so that $f' \leq 0$ on $(c - \delta, c)$ and $f' \geq 0$ on $(c, c + \delta)$ then f has a relative minimum at c.

11. Converse is not true. We can have a relative maximum c so that f' take both positive and negative values on both sides of c, inside any neighborhood of c.

12. **Inequalities:**

$e^x \geq 1 + x$.

$-x \leq \sin(x) \leq x$

if $a > 1$ then $(1 + x)^a \geq 1 + ax$.

13. **Lemma 6.2.11:** Suppose I is an interval, $c \in I$ and f is differentiable at c. Then

(a) if $f'(c) > 0$ there is $\delta > 0$ so that $f(x) > f(c)$ for $x \in I$ so that $c < x < c + \delta$.

(b) if $f'(c) < 0$ there is $\delta > 0$ so that $f(x) > f(c)$ for $x \in I$ so that $c - \delta < x < c$.

Proof:

(a) Since

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) > 0$$

there is a $\delta > 0$ so that $x \in I$ and $|x - c| < \delta$ implies

$$\frac{f(x) - f(c)}{x - c} > 0.$$

If $x > c$ this implies $f(x) - f(c) > 0$.

Proof of (b) is similar. $\square$

14. **Darboux's theorem:** If f is differentiable on $I = [a, b]$ and k is between $f'(a)$ and $f'(b)$ then there is a $c \in (a, b)$ so that $f'(c) = k$.

Proof: Assume $f'(a) < k < f'(b)$ and wet

$$g(x) = kx - f(x).$$

Since g is continuous, it attains a maximum value at some c. Since $g'(a) = k - f'(a)$ the maximum does not occur at a by lemma 6.2.11. Similarly, the maximum does not occur at b. Thus $c \in (a, b)$ and $0 = g'(c) = k - f'(c)$ so $f'(c) = k$. $\square$
15. Even if f is differentiable f' need not be continuous, e.g., $x^2 \sin(1/x)$. So Darboux's theorem does not follow from intermediate value theorem.
16. Darboux's theorem lets us find functions that can't be derivatives of any function.

Section 6.4: Taylor's Theorem

1. Derivatives of order greater than 1: $f'', f''' = f^{(3)}, \ldots$
2. **Defn:** Taylor polynomial of f at x_0:

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 \cdots + \frac{f^{(n)}(x_0)}{n}(x - x_0)^n.$$

3. **Taylor's Theorem:** Let $n \in \mathbb{N}$, $I = [a, b]$, $f : I \to \mathbb{R}$ be such that f and its derivatives up to order n are continuous on I and that $f^{(n=1)}$ exists on (a, b). If $x_0 \in I$ then for any $x \in I$ there is a point c between x_0 and x so that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)}(x - x_0)^{n+1} = P_n(x) + R_n(x).$$

Proof: Fix x and let J be the closed interval with endpoints x and x_0. Define

$$F(t) = f(x) - f(t) - (x - t)f'(t) - \cdots - \frac{(x - t)^n}{n} f^{(n)}(t).$$

Note $F(x) = 0$. Differentiate with respect to t and use product rule :

$$F'(t) = 0 - f'(t) - f'(t) - (x - t)f''(t) - \cdots - \frac{(x - t)^{n-1}}{(n-1)} f^{(n)}(t) - \frac{(x - t)^n}{n} f^{(n+1)}(t).$$

All the terms cancel except the last, so

$$F'(t) = -\frac{(x - t)^n}{n} f^{(n+1)}(t).$$

Define G on J by

$$G(t) = F(t) - \left(\frac{(x - t)}{x - x_0}\right)^{n+1} F(x_0).$$

Then $G(x) = G(x_0) = 0$.

By Rolle's theorem there is a $c \in J$ so that

$$0 = G'(c) = F'(c) = (n + 1) - \frac{(x - c)^n}{(x - x_0)^{n+1}} F(x_0).$$

Thus

$$\begin{aligned} F(x_0) &= \frac{-1}{n+1} - \frac{(x-x_0)^{n+1}}{(x-c)^n} F'(c) \\ &= \frac{1}{n+1} - \frac{(x-x_0)^{n+1}(x-c)^n}{(x-c)^n n} f^{(n+1)}(c) \\ &= -\frac{f^{(n+1)}(c)}{(n+1)}(x-x_0)^{n+1}. \quad \square \end{aligned}$$

4. $e = 1 + \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n} + \ldots$
We assume the derivative of e^x is itself. Then the Taylor polynomial for e^x at $x_0 = 0$ is

$$1 + x + \frac{1}{2}x^2 + \ldots \frac{1}{n}x^n$$

and the remainder is

$$\frac{f^{(n+1)}(c)}{(n+1)}(x-x_0)^{n+1} = e^c(n+1)x^{n+1}$$

for some $c \in [0, x]$. Taking $x = 1$ we see $R_n(1) \to 0$, so

$$e = e^1 = \lim_{n\to\infty}(1 + \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n}).$$

5. $1 - x^2 \leq \cos x$ for all $x \in \mathbb{R}$.
By Taylor' theorem

$$\cos(x) = 1 - x^2/2 + R_2(x)$$

and

$$R_2(x) = \frac{1}{6}f'''(c)x^3 = \frac{1}{6}\sin(c)x^3.$$

If $0 \leq c \leq x \leq \pi$ then $R_2(x) \geq 0$. Similarly if $-\pi \leq x \leq c \leq 0$.
For $|x| \geq \pi$, $1 - x^2 < -3 < \cos(x)$, so this is trivially true.

6. **Theorem 6.4.4:** Suppose I is an interval, x_0 is an interior point of I and $n \geq 2$. Suppose $f', \ldots, f^{(n)}$ are continuous on a neighborhood of x_0 and the first $n-1$ derivatives vanish at x_0.
(i) If n is even and $f^{(n)}(x_0) > 0$, then f has a relative maximum at x_0.
(ii) If n is even and $f^{(n)}(x_0) < 0$, then f has a relative minimum at x_0.
(iii) If n is odd, then f has neither a relative minimum or maximum at x_0.
Proof: see text.

7. **Defn:** A function $f : I \to \mathbb{R}$ is convex if for any $t \in [0, 1]$ and $x, y \in I$, we have

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y).$$

8. **Facts:** convex functions are continuous and left and right derivatives exist at every point.

9. **Theorem 6.4.6:** If f has a second derivative on I then f is convex iff $f''(x) \geq 0$ for all $x \in I$.

Proof:
Suppose f is convex. An exercise shows that

$$f''(a) = \lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}.$$

If f is convex the quotient is ≥ 0 so $f''(a) \geq 0$.

To prove the converse, use Taylor's theorem with $a_1, x_2 \in I$ and $x_0 = (1-t)x_1 + tx_2$ to get

$$f(x) = f(x_0) + f'(x_0)(x_1 - x_0) + \frac{1}{2}f''(c_1)(x_1 - x_0)^2$$

for some c_1 between x_0 and x_1. Similarly,

$$f(x) = f(x_0) + f'(x_0)(x_2 - x_0) + \frac{1}{2}f''(c_2)(x_2 - x_0)^2$$

for some c_2 between x_0 and x_2. If $f'' \geq 0$ then the remainder terms are positive (remember the square), so

$$\begin{aligned}
(1-t)f(x_1) + tf(x_2) &= f(x_0) + f'(x_0)((1-t)x_1 + tx_2 - x_0) \\
&\quad + \frac{1}{2}(1-t)f''(c_1)(x_1 - x_0)^2 \\
&\quad + \frac{1}{2}tf''(c_2)(x_2 - x_0)^2 \\
&= f(x_0) + R \\
&\geq f(x_0) = f((1-t)x_1 + tx_2). \quad \square
\end{aligned}$$

10. **Newton's method:** Let $I = [a, b]$ and let f be twice differentiable on I. Suppose $f(a) < 0 < f(b)$. Suppose there are constants m, M so that

$$0 < m \leq |f'(x)| \text{ and } |f''(x)| \leq M < \infty$$

on I. Them there exists a subinterval J of I containing a zero r of f such that for any $x \in J$ the sequence defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

belongs to J and (x_n) converges to r. Moreover

$$|x_n - r| \leq \frac{M}{2m}|x_n - r|^2$$

for all $n \in \mathbb{N}$.

Proof: see text.

11. Newton's method need not converge if we start too far from root. Iterates can become periodic or chaotic.