MAT 320 Fall 2021, Tuesday, Oct 19, 2021

Section 5.4: Uniform Continuity

1. In the δ-ϵ definition of continuity at x the δ depends on ϵ and x. For example if $f(x) = x^2$, and we fix $\epsilon = 1$, we need to take δ smaller as x gets bigger.
2. **Defn:** $f : A \to \mathbb{R}$ is uniformly continuous on A if for each $\epsilon > 0$ there is a $\delta > 0$ so that if $x, y \in A$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.
3. Uniformly continuous implies continuous, but not conversely.
4. **Non-uniformity Continuity Criteria:** Suppose $f : A \to \mathbb{R}$. TFAE
   (i) f is not uniformly continuous on A.
   (ii) There is $\epsilon_0 > 0$ so that for any $\delta > 0$ there are points $x, y \in A$ with $|x - y| < \delta$ and $|f(x) - f(y)| > \epsilon_0$.
   (iii) There is $\epsilon_0 > 0$ and sequences $(x_n), (y_n)$ so $|x_n - y_n| \to 0$ and $|f(x_n) - f(y_n)| > \epsilon_0$.

Proof: left to reader.

5. **Uniform Continuity Theorem:** If f is continuous on a compact interval, then it is uniformly continuous.

Proof: If f is not uniformly continuous the there is an $\epsilon_0 > 0$ so that for every $n \in \mathbb{N}$ we can choose x_n, y_n so that $|x_n - y_n| < 1/n$ and $|f(x_n) - f(y_n)| > \epsilon_0$. By Bolzano-Weierstrass these sequences have convergent subsequences that both converge to the same point $c \in I$ and hence $f(x_{n_k})$ and $f(y_{m_k})$ must have the same limit. But this is not possible since $|f(x_n) - f(y_n)| > \epsilon_0$ for all n. This contradiction proves f was indeed uniformly continuous. $\square$

6. **Defn:** f is called Lipschitz on A if there is a $K > 0$ so that for all $x, y \in A$

$$|f(x) - f(y)| \leq K|x - y|.$$

7. **Theorem 5.4.5:** If f is Lipschitz on A, it is uniformly continuous.
8. f is called α-Hölder on A if there is a $C > 0$ so that for all $x, y \in A$.

$$|f(x) - f(y)| \leq C|x - y|^\alpha.$$

Such functions are also uniformly continuous.

9. **Theorem 5.4.7:** If f is uniformly continuous on A and (x_n) is Cauchy in A then $(f(x_n))$ is Cauchy in $\mathbb{R}$.

Proof: Given $\epsilon > 0$ choose δ as in the definition of uniformly continuity of f. Then choose $H > 0$ so that $n, m > h$ implies $|x_n - x_m| < \delta$. Thus $n, m > h$ implies $|f(x_n) - f(x_m)| < \delta$. $\square$

10. **Continuous extension Theorem:** If f is uniformly continuous on $I = (a, b)$ iff it can be defined at the endpoints so it becomes continuous on $[a, b]$.

Proof: If there is an extension, the extension is uniformly continuous on $[a, b]$ by Theorem 5.4.3, so f is unif continuous on (a, b).

Conversely, suppose f is uniformly continuous on (a, b). To show it extends to a we need to show $\lim_{x\to A} f$ exists. If (x_n) is any sequence in (a, b) converging to a then it is Cauchy sequence, so $f(x_n)$ is too and hence converges to some $L \in \mathbb{R}$ If (y_n) is any other sequence converging in I to a then $\lim(x_n - y_n) = 0$ so

$$\lim f(y_n) = \lim f(y_n) - f(x_n) - \lim f(x_n) = 0 + L = L.$$

By the sequential criteria for limits of functions, f has limit L at a.

The argument for b is similar.

11. **Theorem 5.4.10:** If f is continuous on a compact interval I and $\epsilon > 0$, then there is a step function s_ϵ on I so that $f(x) - s_\epsilon(x)| < \epsilon$ for all $x \in I$.

Sketch proof.

11. **Theorem 5.4.10:** If f is continuous on a compact interval I and $\epsilon > 0$, then there is a piecewise linear function g_ϵ on I so that $f(x) - g_\epsilon(x)| < \epsilon$ for all $x \in I$.

Sketch proof.

11. **Weierstrass Approximation Theorem:** If f is continuous on a compact interval I and $\epsilon > 0$, then there is a polynomial p_ϵ on I so that $f(x) - p_\epsilon(x)| < \epsilon$ for all $x \in I$.

Numerous proofs, but all use tools we have not developed yet. One common approach is to use Fourier Series to approximate f by finite sums of sines and cosines, and then use power series to approximate each term by polynomials.

Bernstein's proof: given f on $[0, 1]$ f can be approximated by the polynomials

$$B_n(f)(x) = \sum_{k=0}^{n} f(\frac{k}{n}) \frac{n}{(n-k)!k!} x^k (1-x)^{n-k},$$

As n increases, this gets closer and closer to f. Proof is somewhat complicated. See link on class webpage.

Section 5.5: Continuity and Guages – optional section

Omitted. We will do this section later when discussing Appendix C.

Section 5.6: Monotone and Inverse Functions

1. **Defn:**
   f is increasing on A if $x < y$ implies $f(x) \leq f(y)$.
   f is strictly increasing on A if $x < y$ implies $f(x) < f(y)$.
   Similarly for decreasing and strictly decreasing.
   monotone = increasing or decreasing
   strictly monotone = strictly increasing or strictly decreasing
2. Monotone functions need not be continuous
3. **Theorem 5.6.1:** Suppose f is increasing on an interval I and that $c \in I$ is not an endpoint of I. Then

(i) $\lim_{x\to c^-} f = \sup\{f(x) : x < c\}$.
(ii) $\lim_{x\to c^+} f = \inf\{f(x) : x > c\}$.
Proof of (i): Since c is not an endpoint of I, the set on the right is not empty so the supremum L exists and is bounded above by $f(c)$. So for any $\epsilon > 0$, $L - \epsilon$ is not an upper bound, so there is an $y_\epsilon < c$ in I so that $f(y_\epsilon) > L - \epsilon$. If $y \in (y_\epsilon, c)$, then

$$L - \epsilon \le f(y_\epsilon) \le f(y) \le L.$$

Thus

$$|f(y) - L| < \epsilon$$

whenever $|y - c| < |y_\epsilon - c| = \delta(\epsilon)$. This proves $\lim_{x\to c^-} f = L$.

The proof of (ii) is similar. □

4. **Corollary 5.6.2:** Suppose f is increasing on an interval I and that $c \in I$ is not an endpoint of I. Then TFAE
(a) f is continuous at c.
(b) $\lim_{x\to c^-} f = \lim_{x\to c^+} f$
(c) $\sup\{f(x) : x < c\} = \inf\{f(x) : x > c\}$.
5. **Defn:** jump discontinuity. The jump of f at c is

$$j_f(c) = \lim_{x\to c^-} f = \lim_{x\to c^+} f.$$

A jump point is a value c where $j_f(c) > 0$.
6. **Theorem 5.6.3:** Suppose f is increasing on an interval I. Then f is continuous at $c \in I$ iff the jump of f at c is zero.
7. **Theorem 5.6.4:** A monotone function has at mostly countably many jump discontinuities.
Proof: Each jump discontinuity a t x corresponds to a disjoint open interval $I_x = (\lim_{x\to c^-} f, \lim_{x\to c^+} f)$. This interval contains a rational number r_x, and different intervals contain different rationals. This gives injection from the set of jump points into the rationals, a countable set. Thus the number of jump points is countable. □
8. **Continuous Inverse Theorem:** If f is strictly monotone and continuous on an interval I, then f has an inverse g that is strictly monotone and continuous.
Proof: Enough to consider f increasing.

By Theorem 5.3.10, $J = f(I)$ is an interval. Since strictly monotone implies injective, f has an inverse $g : J \to I$.

Easy to see that g is strictly increasing: if $x, y \in J$ and $x < y$ Then $f(g(x)) < f(g(y))$ so $g(x) < g(y)$ since f is increasing.

If g were discontinuous, then it must have a jump discontinuity at some point $c \in J$. This means there is a value between $\lim_{x\to c^-} g$ and $\lim_{x\to c^+} g$ which is not in the image of g. Thus $g(J) = I$ is not an interval, a contradiction. □

9. nth roots exist for $x \ge 0$ if n is even.
10. nth roots exist for $x \in \mathbb{R}$ if n is even.
11. positive rational powers x^r exist for $x \ge 0$.
12. all rational powers x^r exist for $x > 0$.