MAT 319 & 320 Fall 2021, Lecture 14, Thursday, Oct 7, 2021

Section 5.1: Some extensions of the limit concept

1. **Defn:** Let $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ and $c \in A$. We say f is **continuous** at c if for any $\epsilon > 0$ there is a $\delta > 0$ so that $x \in A$, $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$.
   For cluster points of A, this is the same as saying $\lim_{x \to c} f(x) = f(c)$.
   For isolated points, f is always continuous.
2. **Theorem 5.1.2:** f is continuous at c iff given any $\epsilon > 0$ there is a $\delta > 0$ so that

$$f(V_\delta(c)) \subset V_\epsilon(f(c)).$$

Proof: obvious from definitions.
3. **Sequential criterion for continuity:** f is continuous at $c \in A$ iff for every sequence $x_n \to c$ we have $f(x_n) \to f(c)$.
3. **Discontinuity criterion:** f is discontinuous at $c \in A$ iff there is a sequence $x_n \to c$ so that $f(x_n) \not\to f(c)$.

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4. $\sin(1/x)$ is discontinuous at 0. $x_n = 1/\pi n$, $y_n = -1/\pi n$.
5. We say f is continuous on a set $B \subset A$ if f is continuous at every point of B.
6. Examples:
   constant functions
   identity function
   powers, polynomials
   rational functions, except at poles
   Dirichlet function, discontinuous everywhere
   Thomae's function, discontinuous exactly at rationals
7. If c is a cluster point of A but not in A, then a function f on A can be defined at c to be continuous there iff f has a limit as $x \to c$ in A.

Section 5.2: Combinations of Continuous Functions

1. **Theorem 5.2.1:** Let $A \subset \mathbb{R}$, $f, g : A \to \mathbb{R}$, $b \in \mathbb{R}$ and $c \in A$. If f, g are continuous at c then
   (a) $f + g$, $f - b$, fg, bf are continuous at c.
   (b) if also $g(x) \neq 0$ for all $x \in A$, then f/g is continuous at c.

Proof: If c is isolated this is automatic.
If c is a cluster point of A then

$$\lim_{x\to c}(f+g)(x) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = f(c) + g(c) = (f_g)(c),$$

so $f + g$ is continuous. Other parts are similar. □

2. **Theorem 5.2.2:** Let $A \subset \mathbb{R}$, $f, g : A \to \mathbb{R}$ are continuous on A and $b \in \mathbb{R}$.
   (a) $f + g$, $f - b$, fg, bf are continuous at c.
   (b) if also $g(x) \neq 0$ for all $x \in A$, then f/g is continuous at c.
3. Examples
   polynomials are continuous on all of $\mathbb{R}$.
   rational functions are continuous where they are defined
   $\sin(x)$ continuous. This function is not officially defined yet, but you know it from calculus. We will prove

$$|\sin x| \leq |x|$$

$$\sin x - \sin y = 2\sin(\frac{1}{2}(x-y))\cos(\frac{1}{2}(x+y)).$$

These imply

$$|\sin x - \sin y| \leq |x - y|.$$

So we can take $\delta = \epsilon$ in defn of continuity.
   $\cos(x)$ continuous. Similar, but we use

$$\cos x - \cos y = -2\sin(\frac{1}{2}(x+y))\sin(\frac{1}{2}(x-y)).$$

These imply

$$|\sin x - \sin y| \leq |x - y|.$$

So we can take $\delta = \epsilon$ in defn of continuity.

4. **Theorem 5.2.4:** If $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ let $|f|(x) = |f(x)|$. If f is continuous at c, so is $|f|$.

Proof: Note that

$$||f|(x) - |f|(y)| = ||f(x)| - |f(y|| \leq |f(x) - f(y)|.$$

So if $|x-y| < \delta$ implies $|f(x)-f(y)| < \epsilon$, then it also implies $||f|(x)-|f|(y)| < \epsilon$. □
   Note that f can be discontinuous, but $|f|$ is continuous.

5. **Theorem 5.2.5:** If $A \subset \mathbb{R}$ and $f : A \to [0, \infty)$. Define $\sqrt{f}(x) = \sqrt{f(x)}$.
   (a) if f is continuous at c, so is $\sqrt{f}$.

Proof: If $0 < \delta < c$ and $|x - c| < \delta$, then $x > 0$. Thus

$$\sqrt{x} - \sqrt{c} = (\sqrt{x} - \sqrt{c})\frac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{c}} = \frac{x - c}{\sqrt{x} + \sqrt{c}}.$$

Thus

$$|\sqrt{x} - \sqrt{c}| \leq \frac{|x - c|}{\sqrt{x} + \sqrt{c}} \leq \frac{|x - c|}{\sqrt{c}}.$$

So given $\epsilon > 0$ choose $\delta = \min(c, \sqrt{c}\epsilon)$. Then $|x - c| < \delta$ implies

$$|\sqrt{x} - \sqrt{c}| \leq \frac{\sqrt{c}\epsilon}{\sqrt{c}} = \epsilon. \quad \square$$

6. **Theorem 5.2.7:** Let $A, B \subset \mathbb{R}$ and let $f : A \to B$ and $g : B \to \mathbb{R}$. If f is continuous at c and g is continuous at $f(c)$ then $g \circ f$ is continuous at c.
Proof: Let W be a ϵ-neighborhood of $g(c)$. Since g is continuous at $b = f(c)$ there is a δ-neighborhood V of b so that $g(V) \subset W$. Since f is continuous at c, there is a γ-neighborhood U of c so that $f(U) \subset V$. Hence $g(f(U)) \subset g(V) \subset W$. Thus $g \circ f$ is continuous at c. $\square$
7. **Theorem 5.2.7:** Let $A, B \subset \mathbb{R}$ and let $f : A \to B$ and $g : B \to \mathbb{R}$. If f is continuous on A and g is continuous on B then $g \circ f$ is continuous on B.
8. Examples:
   $g(x) = |x|$, and any f. Then $|f|$ is continuous.
   $g(x) = \exp(x)$, $f(x) = \frac{1}{2}\log x$, $A = (0, \infty)$. then $g \circ f(x) = \sqrt{x}$ is continuous (if we can define exp and log. This can be done using integration.

Section 5.3: Continuous functions on intervals

1. **Defn:** A function $f : A \to \mathbb{R}$ is bounded if $f(A)$ is a bounded set, i.e., if there is $M > 0$ so that $|f(x)| \leq M$ for all $x \in A$.
2. **Boundedness Theorem:** If $I = [a, b]$ is a closed bounded interval and $f : I \to \mathbb{R}$ is continuous then f is bounded on I.
Proof: Suppose f is not bounded on I. Then for any $n \in \mathbb{N}$ we may choose $x_n \in I$ so that $|f(x_n)| > n$. By the Bolzano-Weierstrass theorem there is a convergent subsequence (x_{n_k}) and by Theorem 3.2.6 the limit point x is in I. But then $f(x_{n_k}) \to f(x)$ and by Theorem 3.2.2 this sequence is bounded, a contradiction. $\square$
3. **Examples:**
   Can fail if I is not bounded, or not closed.
   Can fail if f is not continuous
4. **Defn:** $f : A \to \mathbb{R}$ has an absolute (global) maximum on A if there is a point $y \in A$ so that $f(x) \leq f(y)$ for all $x \in A$.
   Similar for absolute minimum.
   Need not be unique. $\sin(x)$.

5. Closed bounded interval is called a compact interval. Special case of a compact set.
6. **Maximum-Minimum Theorem:** A continuous function f on a closed, bounded interval I has an absolute maximum and minimum.
Proof: First we consider maximums. Choose a sequence $(x_n) \subset A$ so that $f(x_n) \to s = \sup f(A)$. By Bolzano-Weierstrass there is a convergent subsequence $(x_{n_k}) \to x \in A$ and $f(x_{n_k}) \to f(x)$. But $\lim f(x_{n_k}) = \lim f(x_n) = s$, so $f(x) = s$. Thus f has an absolute maximum.

For the minimum, not that $g = -f$ is continuous on I so has an absolute maximum $g(y)$ by above argument. Thus for all $x \in I$ we have $f(x) = -g(x) \geq -g(y) = f(y)$ and so $f(y)$ is an absolute minimum of f on I. □-
7. **Location of Roots Theorem:** Suppose f is continuous on $I = [a, b]$. If $f(a) < 0 < f(b)$, then there is a $c \in (a, b)$ so that $f(c) = 0$.
Proof: We will define nested, decreasing intervals $I_n = [a_n, b_n]$. Let $I_1 = I$ so $a_1 = 1$, $b_1 = b$. Let $m_1 = (a_1 + b_1)/2$ be the midpoint of I_1. If $f(m_1) = 0$ the theorem is proved, so assume it is not 0. If $f(m_1) > 0$, let $a_2 = a_1$ and $b_2 = m$. $f(m_1) < 0$, let $a_2 = m$ and $b_2 = b_1$.

Proceeding inductively either some m_n is a zero of f, or we construct nested, decreasing sequence $I_1 \supset I_2 \supset \dots$ whose lengths are $|b - a|2^{-n} \to 0$. The Nested Intervals Property (Thm 2.5.2) says there is a unique point c in the intersection and

$$c = \lim a_n = \lim b_n,$$

so by continuity of f

$$f(c) = \lim f(a_n) = \lim f(b_n).$$

But

$$\lim f(a_n) \leq 0, \quad \lim f(b_n) \geq 0,$$

so we must have

$$f(c) = \lim f(a_n) = \lim f(b_n) = 0. \quad \square$$

8. **Bolzano's intermediate Value Theorem:** If f is continuous on an interval I and $k \in \mathbb{R}$ is such that $f(a) < k < f(b)$ then there is a $c \in (a, b)$ so that $f(c) = k$.
Proof: Apply previous result to $f(x) - k$.
9. **Corollary 5.3.8:** If f is continuous on interval I and $\inf_I f \leq k \leq \sup_I f$ then there is a $c \in I$ with $f(c) = k$.
10. **Theorem 5.3.9:** If I is a compact interval and f is continuous on I, then $f(I) = \{f(x) : x \in I\}$ is a compact interval.
Proof: By the previous Corollary $f(I)$ contains every point in $J = [\sup_I f, \inf_I f]$ and by definition of sup and inf it contains no points outside J. Thus $f(I) = J$, and J is bounded by the Boundedness Theorem. □
11. **Notes:**

Max and min don't have to be attained at endpoints.

Image of an open interval need not be open.

Image of unbounded, closed interval need not be closed.

12. **Theorem 5.3.10:** If f is continues on an interval I then $f(I)$ is also an interval.

Proof: If $\alpha = f(a) < \beta = f(b)$ are two points in $f(I)$ then any value between them is also in $f(I)$ by the intermediate value theorem. Thus $[\alpha, \beta] \subset f(I)$. By Theorem 2.5.1 $f(I)$ is an interval. $\square$