320 Fall 2021, Tuesday, Nov 9, 2021

Section 8.1: Pointwise and Uniform Convergence

1. **Defn:** A sequence of functions (f_n) is a choice of function $f_n : A \to \mathbb{R}$ for each $n \in \mathbb{Z}$. The domain A should be the same for every n.
2. We say f_n converges pointwise to f on A if $f_n(x) \to f(x)$ for every $x \in A$.
3. Examples:
   $f_n(x) = x/n$ on $\mathbb{R}$
   $f_n(x) = x^n$ on $[0, 1]$.
4. **Defn:** We say f_n converges to f uniformly on A if for all $\epsilon > 0$ there is a K so that $n \geq K$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \in A$.
5. Uniform convergence implies pointwise convergence, but not conversely. We say uniform convergence is "stronger" than pointwise convergence.
6. **Lemma 8.1.5:** f_n does not converge uniformly on A to f iff there is some $\epsilon_0 > 0$ and a subsequence f_{n_k} and a sequence (x_k) so that

$$|f_{n_k}(x_k) - f(x_k)| \geq \epsilon_0.$$

7. **Defn:** We say $\varphi : A \to \mathbb{R}$ is bounded if $\varphi(A)$ is a bounded set, i.e., $|\varphi(x)| \leq M$ for some M and all $x \in A$. Define

$$\|\varphi\|_A = \sup\{|\varphi(x)| : x \in A\}.$$

8. Bounded functions on a set form a vector space. This is a norm on that vector space.
9. **Lemma 8.1.8:** A sequence f_n converges uniformly to f on A iff

$$\|f_n - f\|_A \to 0.$$

Proof: If $f_n \to f$ uniformly then for all ϵ there is a K so that $n \geq K$ implies

$$|f_n - f\|_A = \sup\{|f_n(x) - f(x) : x \in A\} \to 0.$$

Conversely, if $|f_n - f\|_A \to 0$, then for all ϵ there is a K so that $n \geq K$ implies $|f_n - f\|_A < \epsilon$, which is the same as $|f_n(x) - f(x)| < \epsilon$ for all $x \in A$.
10. **Cauchy Criterion for Uniform Convergence:** Suppose f_n is sequence of bounded functions on A. Then f_n converges uniformly on A to a bounded function f iff for all $\epsilon > 0$ there is a H so that for all $n, m \geq H$ we have $\|f_n - f_m\|_A < \epsilon$.
Proof: If $f_n \to$ uniformly then for any $\epsilon > 0$ there is a K so that $n \geq K$ implies $\|f - f_n\|_A < \epsilon/2$, so $n, m \geq K$ implies

$$\|f_m - f_n\|_A \|f - f_n\|_A + \|f - f_m\|_A < \epsilon/2 + \epsilon/2 < \epsilon.$$

Conversely, if the Cauchy condition holds for f_n, then for each $x \in A$ we have

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_A,$$

so $(f_n(x))$ is a Cauchy sequence of real numbers and hence converge so some limit we call $f(x)$. Since

$$|f(x) - f_m(x)| \leq \limsup_n |f_n(x) - f_m(x)| \leq \epsilon$$

if $m \geq H$, we see that $f_m \to f$ uniformly.

Section 8.2: Interchange of Limits

1. Questions:
   Is a limit of continuous functions continuous? x^n
   Is a limit of differentiable functions differentiable?
   Is a limit of Riemann integrable functions Riemann integrable? Sliding tent.

2. **Theorem 8.2.2:** If (f_n) is a sequence of continuous function converging uniformly on A to f, then f is continuous on A.
Proof: Given $\epsilon > 0$ there is a H so that $n \geq H$ implies $|f_n(x) - f(x)| < \epsilon/3$ for all $x \in A$. Also, given $c \in A$ there exists $\delta > 0$ so that $|x - c| < \delta$ implies $|f_n(x) - f_n(c)| < \epsilon/3$. Thus for $|x - c| < \delta$, we have

$$|f(x) - f(c)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)| \leq \epsilon.$$

3. Partial sums of $\sum_{n=1}^{\infty} 2^{-n} \cos(3^n x)$ show that differentiable functions can converge uniformly to a nowhere differentiable function.

4. **Theorem 8.2.3:** Let $J = [a, b] \subset \mathbb{R}$ be a bounded interval and (f_n) a sequence of functions on J. Suppose there is a $x_0 \in J$ so that $f_n(x_0)$ converges and that f_n' converge uniformly to g on J. Then f_n converges uniformly to a a differentiable function f on J. so that $f' = g$.

Proof: Take some $x \in J$. Apply the mean value theorem to $f_n - f_m$ to find a point y between x_0 and x so that

$$f_m'(y) - f_n'(y) = f_m(x_0) - f_n(x_0) + (x - x_0)(f_m(x) - f_n(x)).$$

Thus

$$\|f_m'(y) - f_n'(y)\|_J \leq |f_m(x_0) - f_n(x_0)| + |b - a| \cdot \|f_m(x) - f_n(x)\|_J.$$

Hence $\{f_n\}$ is Cauchy and therefore convergent. Thus it has a continuous limit f.

Take $c \in J$. To prove $f'(c)$ exists, apply the mean value theorem between x and c to find a z between them so that

$$f_m(x) - f_n(x) - (f_m(c) - f_n(c)) = (x - c)(f_m(z) - f_n(z)).$$

$$|\frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c}| \leq \|f_m(z) - f_n(z)\|_J.$$

For any $\epsilon > 0$ there is an H so that $n, m \geq H$ imply

$$|\frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c}| \leq \epsilon.$$

Take the limit over m
$$|\frac{f(x)-f(c)}{x-c}-\frac{f_n(x)-f_n(c)}{x-c}|\leq\epsilon.$$
Since $g(c)=\lim_n f_n'(c)$, there is an N so that $n\geq N$ implies
$$|g(c)-f_n'(c)|<\epsilon.$$
Let $K=\max(H,N)$. Since $f_K'(c)$ exists, there is a $\delta>0$ so that $0<|x-c|<\delta$ implies
$$|\frac{f_K(x)-f_K(c)}{x-c)}-f_K'(c)<\epsilon.$$
Hence if $0<|x-c|<\delta$, we have
$$|\frac{f(x)-f(c)}{x-c}-g(c)|<3\epsilon.$$
Hence $f'(c)=g(c)$. □

5. **Theorem 8.2.4:** If (f_n) are Riemann integrable functions converging uniformly to f then f is Riemann integrable and
$$\int_a^b f=\lim_n\int_a^b f_n.$$
Proof: Given any $\epsilon>0$ there is an N so that $n>N$ implies
$$\alpha(x)=f_n(x)-\epsilon\leq f(x)\leq\omega(x)=f_n(x)+\epsilon$$
Both α and ω are Riemann integrable and
$$\int_a^n\omega-\alpha=\epsilon(b-a).$$
Thus by the squeeze theorem f is Riemann integrable and for all n
$$\int_a^b f_n(x)-\epsilon(b-a)\leq\int_a^b f(x)\leq\int_a^b f_n(x)+\epsilon(b-a)$$
$$|\int_a^b f-\int_a^b f_n|<\epsilon.$$
Thus $\int f_n\to\int f$. □

Both α and ω are Riemann integrable and

6. **Theorem 8.2.5:** Suppose (f_n) are Riemann integrable functions on $[a,b]$ converging pointwise to a to a Riemann integrable function f. Suppose also that there exists a B so that $|f_n(x)|\leq B$ for all n and all $x\in[a,b]$. Then $\int f_n\to\int f$.

Proof: see link on class webpage.

6. **Dini's Theorem:** Suppose (f_n) is a monotone sequence of continuous function on $I=[a,b]$ that converges pointwise to f. Then $f_n\to f$ uniformly.

Proof: We assume $f_1\geq f_2\geq\dots$. Let $f_n=f_n-f\geq 0$. It is enough to show $g_n\to 0$ uniformly.

Given $\epsilon > 0$ and $t \in I$ there is a M so that $0 \leq g_m(t) < \epsilon$. Since g_m is continuous there is a $\delta(t) > 0$ so that $|x - t| < \delta(t)$ implies $|g_m(t) - g_m(x)| < \epsilon$. Take $\delta(t)$ as a guage on I and let $\mathcal{P}$ be a δ-fine tagged partition. Let $M = \max(m_{t_1}, \ldots m_{t_n})$. If $m > M$ and $x \in I$ then there is an index k with $|x - t_k| < \delta(t_k)$ so

$$0 \leq g_m(x) \leq g_{m_k}(x) < \epsilon.$$

Thus $g_n \to 0$ uniformly. $\square$

Planning to skip Sections 8.3 and 8.4 in text.