320 Fall 2021, Tuesday, Nov 30, 2021

Section 11.1: Open and closed sets

1. **Defn:** a neighborhood of a point $x \in \mathbb{R}$ is any set V that contains an open interval around x, i.e., $(x-\delta, x+\delta) \subset V$ for some $\delta > 0$.
2. **Defn:** A set $G \subset \mathbb{R}$ is open if for each $x \in G$ there is a neighborhood V of x with $V \subset G$.
3. **Defn:** A set is closed if its complement is open.
4. **Examples:**
   every open interval
   $[0,1]$ is closed
   empty set
   $\mathbb{R}$ is both open or closed
   $\{\frac{1}{n}\} \cup \{0\}$ is closed
   rationals are neither
5. The difference between doors and sets.
6. **Open Set Properties:**
   (a) any union of open sets is open.
   (b) a finite intersection of open sets is open.

Proof:
(a) If $x \in G = \cup_\lambda G_\lambda$ then x is in some G_λ. Thus it has a neighborhood in G_λ and hence in G. $\square$

(b) Suppose $x \in \cap_1^n G_k$. There are $\delta_k > 0$ so that $(x - \delta_k x + \delta_k) \subset G_k$. So if $\delta = \min \delta_k > 0$ (positive since finite collection), then $(x - \delta x + \delta) \subset G_k$ for all k and hence $(x - \delta x + \delta) \subset \cap_k G_k$. $\square$

7. **Closed Set Properties:**
   (a) any intersection of closed sets is closed.
   (b) a finite union of closed sets is closed.

Proof:
(a) follows from the previous result and

$$\mathbb{R} \setminus \cap_\lambda F_\lambda = \cup_\lambda (\mathbb{R} \setminus F_\lambda).$$

(b) Follows from the previous result and

$$\mathbb{R} \setminus \cup_\lambda F_\lambda = \cap_\lambda (\mathbb{R} \setminus F_\lambda).$$

8. The intersection of countable many open sets need not be open. For example the irrationals.
   A countable intersection of open sets is called G_δ.
   A countable union of closed sets is called F_σ.
   A countable union of G_δ sets is called $G_{\delta\sigma}$

$\vdots$

All these categories are different, they become larger and larger collections of sets.

This is the beginning of the Borel hierarchy of sets.

Not every set is in the hierarchy. There are non-Borel sets.

9. **Theorem 11.1.7:** A set $F \subset \mathbb{R}$ is closed iff every convergent sequence in F has its limit in F.

Proof: Suppose $(x_n) \subset F$ and $x_n \to x$. We claim $x \in F$. If not, x is in the open complement of F so there is an interval $(x-\delta, x+\delta) \subset I \subset F^c$. But then $|x_n - x| > \delta$ for all n, a contradiction. Hence $x \in F$.

Conversely, if F is not closed then f^c is not open, so there is a point $x \in F^c$ so that every interval $(x - \frac{1}{n}, x + \frac{1}{n})$ contains a point $x_n \in F$. Thus $(x_n) \subset F$ and $x_n \to x \notin F$. $\square$

10. **Theorem 11.1.8:** A set is closed iff it contains all its cluster points.

Proof: left to reader.

11. **Theorem 11.1.9:** A set in $\mathbb{R}$ is open iff it is a countable union of disjoint intervals.

Proof: We already know that any union of open intervals in open.

Conversely, suppose G is non-empty and open. For $x \in G$ let $\mathcal{C}_x$ be the collection of open intervals I so that $x \in I \subset G$. Since G is open this is non-empty and its union I_x is open and is contained in G.

If $y \in G$ is another point and $I_x \cap I_y \neq \emptyset$ then $I_x \cup I_y$ is an open interval containing x and inside G so is in $\mathcal{C}_x$. Hence $I_x \cup I_y \subset I_x$ and thus $I_y \subset I_x$.

But exchanging the roles of x and y proves $I_x \subset I_y$. Thus either I_x and I_y are disjoint or they are the same.

Each I_x contains a rational point and disjoint intervals cannot contain the same point, so there are only countable many different sets I_x that can occur. Thus G is a union of countably many disjoint open intervals. $\square$

12. **The Middle Thirds Cantor set:**

Define

Is closed.

Has zero length.

Ternary expansion

Contains no intervals (otherwise not null set)

Is uncountable.

13. There are many other types of Cantor set: closed, uncountable, nowhere dense, every point is a cluster point.

13. There are Cantor sets of positive length (requires some measure theory to prove this precisely).

Section 11.2: Compact sets

1. **Defn:** an open cover of a set A is any collection of open sets whose union contains A.

A finite subcover is a finite subcollection whose union also covers.

2. **Defn:** A set is compact iff every open cover contains a finite subcover.

This makes sense in arbitrary topological spaces. Very general, very useful concept.

3. **Theorem 11.2.4:** A compact set $K \subset \mathbb{R}$ must be closed an bounded.

Proof: If it is not bounded then $(-n, n)$ is an open cover of K with no finite subcover.

If it is not closed an $x \in K^c$ is a limit point then $\{y : |y - x| > 1/n\}$ is an open cover of K with no finite subcover. $\square$

4. **Heine-Borel Theorem:** A set $K \subset \mathbb{R}$ is compact iff it is closed and bounded.

Proof: we already proved these conditions are necessary.

Conversely, suppose K is closed and bounded and let $\{G_\alpha\}$ is any open cover of K.

First do the case when $K = I$ is a closed bounded interval. For each $x \in I$ choose $\delta = \delta(x)$ so that $(x - \delta, x + \delta)$ is inside some element G_α. This is a guage, so by Theorem 5.5.5, there is a δ-fine partition. If for the kth partition element we take the element of G_α of the cover that contains it, we get a finite subcover of I.

If K is not an interval, but $K \subset I = [-n, n]$, then add K^c to the collection of open sets. This covers I, hence has a finite subcover. Since K^c doesn't cover any point of K, the finitely many sets chosen from $\{G_\alpha\}$ must cover K. $\square$

Theorem 11.2.6: A set $K \subset \mathbb{R}$ is compact iff every sequence in K has a convergent subsequence.

Proof: Heine-Borel + Bolzano-Weierstrass.

6. Not true in all settings. In an infinite dimensional vector space, a set can be closed and bounded but not compact.

Section 11.3: Continuous functions

1. **Lemma 11.3.1:** A function $f : A \to \mathbb{R}$ is continuous at $c \in A$ if for every neighborhood U of $f(c)$ there is a neighborhood V of c so that $f(V) \subset U$.

Proof: If this conditions holds and $\epsilon > 0$, then take $U = (f(c) - \epsilon, f(c) + \epsilon)$ and take V so $f(V) \subset U$. Since V is a neighborhood there is a $\delta < 0$ so $I = (c - \delta, c + \delta) \subset V$, so $f(I) \subset U$. Thus $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$, i.e., f is continuous at c.

Conversely, if f is continuous at c then for any neighborhood of $f(c)$ choose ϵ so $(f(c) - \epsilon, f(c) + \epsilon) \subset U$, and take $V = (c - \delta, c + \delta)$. $\square$

2. **Defn:** $B \subset A$ is open in A if $B = A \cap U$ for some open $U \subset \mathbb{R}$.

3. **Theorem 11.3.2:** $f : A \to \mathbb{R}$ is continuous every where on A iff for every open set $G \subset \mathbb{R}$ we have that $f^{-1}(G)$ is open in A.

Proof:

First assume f is continuous and G is open. If $x \in f^{-1}(G)$ then G is a neighborhood of $f(x)$, so there is an open neighborhood V of x so that $f(V) \subset G$ or $V \subset f^{-1}(G)$. Then the union of these open neighborhoods is an open set whose intersection with A is $f^{-1}(G)$.

Conversely, assume the condition holds. Let $c \in A$. For any open neighborhood G of $f(c)$ there is an open set H with $H \cap A = f^{-1}(G)$. Then $c \in H$. If $x \in H \cap A$ then $f(x) \in G$. Thus H is a neighborhood of c that maps into G, so f is continuous at c. Hence f is continuous at every point of A. $\square$

4. **Corollary 11.3.3:** A function $f : \mathbb{R} \to \mathbb{R}$ is continuous iff the inverse image of every open set is open.
5. In topology, a function is defined to be continuous if the inverse image of every open set is open.
6. The continuous image of an open set need not be open, e.g., $\sin(\mathbb{R})$.
7. **Theorem 11.3.4:** If $K \subset \mathbb{R}$ is compact and $f : K \to \mathbb{R}$ is continuous, then $f(K)$ is compact.

Proof: Let $\{G_\alpha\}$ be an open cover of $f(K)$. Then $\{f^{-1}(G_\alpha)\}$ is an open cover of K, so has a finite subcover. The corresponding G_α give an finite subcover of $f(K)$. Therefore every finite subcover of $f(K)$ has a finite subcover, so $f(K)$ is finite. $\square$

8. **Theorem 11.3.6:** If $K \subset \mathbb{R}$ is compact and $f : K \to \mathbb{R}$ is continuous and injective, then f^{-1} is continuous.

Proof: By the last result $f(K)$ is compact. If G is open in $\mathbb{R}$ then $E = K \subset G$ is closed and bounded, hence compact. Hence $f(E) \subset f(K)$. Since f is 1-to-1,

$$f(G \cap K) = f(K) \setminus f(K \setminus G) f(K) \cap f(E)^c.$$

Thus the inverse image under f^{-1} of any open set is open in $f(K)$. Thus f^{-1} is continuous. $\square$

Section 11.4: Metric Spaces

1. Metric spaces have a notion of distance between points, but not of order, addition, multiplication,... unless we assume this as extra.
2. **Defn:** A metric space is a set S with a function $d : S \times S \to [0, \infty)$ so that
   (1) $d(x, y) \geq 0$ for all x, y,
   (2) $d(x, y) = 0$ iff $x = y$,
   (3) $d(x, y) = d(y, x)$,
   (4) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in S$.
3. **Examples:**
   $\mathbb{R}$ with $d(x, y) = |x - y|$
   Any subset of $\mathbb{R}$ with the same metric.

Given an positive, continuous function f on reals, we can define for $x \leq y$

$$d(x,y) = \int_x^y f.$$

and $d(y,x) = d(x,y)$ if $y < x$.

$\mathbb{R}^n$ with $d(x,y) = \sqrt{\sum_1^n (x_k - y_k)^2}$.

$\mathbb{R}^n$ with $d_1(x,y) = \sum_1^n |x_k - y_k|$

$\mathbb{R}^n$ with $d_\infty(x,y) = \sup_k |x_k - y_k|$

Discrete metric on any set $d(x,y) = 1$ unless $x = y$; then $d(x,x) = 0$.

All bounded functions $f : A \to \mathbb{R}$ with $d(f,g) = \sup_A |f(x) - g(x)|$.

All continuous functions $f : [0,1] \to \mathbb{R}$ with $d(f,g) = \sup_A |f(x) - g(x)|$.

All continuous functions $f : [0,1] \to \mathbb{R}$ with $d(f,g) = \int_0^1 |f(x) - g(x)|$.

All continuous functions $f : [0,1] \to \mathbb{R}$ with $d(f,g) = \left(\int_0^1 |f(x) - g(x)|^2\right)^{1/2}$.

ℓ^∞ = all bounded sequences (x_n) with $d((x_n),(y_n)) = \sup_n |x_n - y_n|$

c_0 = all bounded sequences $(x_n) \to 0$ with $d((x_n),(y_n)) = \sup_n |x_n - y_n|$

ℓ^1 = all absolutely convergent sequences $(x_n) \to 0$ with $d((x_n),(y_n)) = \sum_n |x_n - y_n|$

On subsets of any finite set S, $d(A,B)$ is number of elements in $\Delta(A,B) = (A \cup B) \setminus (A \cap B)$ (number of elements in one set but not the other).

Set of all finite strings of symbols from an alphabet A $a_1, a_2, a_3, \ldots a_n$ with $d((a_k), b(k)) = 1/n$ where $n = inf\{N : a_k = b_k, k = 1, \ldots, N\}$.

Edit distance between words = minimal number of replacements. deletions and additions needed to convert one word to another, $d(\text{ball}, \text{ballon}) = 2$ to another, $d(\text{ball}, \text{call}) = 1$

Graph distance. In graph theory, the distance between two vertices of a connected graph is the fewest number of edges needed to connect them.

Many metrics between shapes. Hausdorff metric between compact sets E and F is infimum of $\epsilon > 0$ so that

$$E \subset \{y : \mathrm{dist(y,F)} < \epsilon\}$$

and

$$F \subset \{y : \mathrm{dist(y,E)} < \epsilon\}$$

where

$$\mathrm{dist(x,E)} = \inf_{\mathrm{y \in E}} \{|\mathrm{x - y}|\}.$$

4. **Defn:** an ϵ-neighborhood of a point $x \in S$ is $V_\epsilon(x) = \{y \in S : d(x,y) < \epsilon\}$

Often denoted $B(x,\epsilon)$ to denote the ball of radius ϵ around x.

Given this we can define open sets, closed sets, compact sets.

5. **Defn:** a sequence (x_n) in a metric space S converges to $x \in S$ if $d(x_n, x) \to 0$. Equivalently, for every neighborhood V of x, $x_n \in V$ for all n large enough.

6. **Defn:** a sequence in S is Cauchy if for all $\epsilon > 0$ there is an H so that $n, m > H$ implies $d(x_n, x_m) < \epsilon$.

6. **Defn:** a metric space is called complete if every Cauchy sequence converges.

7. $C[0,1]$ is complete with the supremum metric.
8. **Defn:** A set G is open in S if it contains a neighborhood of each of its points.
A set is closed if its complement is open.
A set is compact if every open cover has a finite subcover.
A mapping $f : S_1 \to S_2$ between metric spaces is continuous at a point $c \in S_1$ if for every neighborhood $U \subset S_2$ of $f(c)$ there is a a neighborhood $V \subset S_1$ so that $f(V) \subset U$.
9. **Theorem 11.4.11:** a map between metric spaces is continuous iff the preimage of every open set is open.
10. **Theorem 11.4.12:** If $f : S_1 \to S_2$ is continuous, then the image of any compact set is compact.
11. A subset K of metric space is compact iff every sequence in K has a convergent subsequence.
12. A compact set of a metric space is closed and bounded (contained in a ball of finite radius). The converse is not true. Thus the Heine-Borel theorem is not true in general metric spaces.
The set of functions in $C[0,1]$ with $I \sup |f| \leq 1$ is closed and bounded, but is not compact. Consider $\{x^n\}$ that has no uniformly convergent subsequence.
The closed unit ball of a normed vector space is compact iff the space is finite dimensional.