320 Fall 2021, Tuesday, Nov 2, 2021

Section 7.2: Riemann Integrable Functions

1. **7.2.1 Cauchy Criterion:** A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable iff for every ϵ there is a $\eta > 0$ so that if $\mathcal{P}$, $\mathcal{Q}$ are tagged partitions with $\|\mathcal{P}\| < \eta$ and $\|\mathcal{Q}\| < \eta$, then

$$|S(f, \mathcal{P})S(f, \mathcal{Q})| < \epsilon.$$

Proof:

On direction is easy. If f is Riemann integrable, and $\epsilon > 0$, choose δ so that $\|\mathcal{P}\| < \delta$ implies the Riemann sums are within $\epsilon/2$ of $\int f$, and then any two of them are with ϵ of each other.

Conversely, for each $n \in \mathbb{N}$, choose $\delta_1 > \delta_2 > \ldots$ so that $\|\mathcal{P}\|, \|\mathcal{Q}\| < \delta_n$ implies the corresponding Riemann sums are within $1/n$ of each other. Choose a sequence of partitions $\mathcal{P}_n$ with $\|\mathcal{P}_n\| < \delta_n$. Then the Riemann sums $\{s_m\}$ form a Cauchy sequence and hence converge to some limit A and

$$\|S(f, \mathcal{P}_n) - A\| \le \delta_n.$$

Then for any tagged partition with $\|\mathcal{P}\| < \delta_n$

$$|S(f, \mathcal{P}) - A| \le +|S(f, \mathcal{P}_n) - A| + |S(f, \mathcal{P}_n) - S(f, \mathcal{P})| \le \epsilon. \quad \square$$

2. **7.2.3 Squeeze Theorem:** $f : [a, b] \to \mathbb{R}$ is Riemann integrable iff for every $\epsilon > 0$ there exist integrable functions α_ϵ and ω_ϵ so that

$$\alpha_\epsilon(x) \le f(x) \le \omega_\epsilon(x)$$

for all $x \in [a, b]$ and

$$\int \omega_\epsilon - \alpha_\epsilon < \epsilon.$$

Proof:

If f is integrable take $\alpha_\epsilon = \omega_\epsilon = f$.

For the converse, assume $\epsilon > 0$ and α, ω are as in the theorem. Choose a $\delta > 0$ so that $\|\mathcal{P}\| < \delta$ implies

$$|S(\alpha, \mathcal{P}) - \int \alpha| \le \epsilon,$$

$$|S(\omega, \mathcal{P}) - \int \omega| \le \epsilon.$$

Since

$$S(\alpha, \mathcal{P}) \le S(f, \mathcal{P}) \le S(\omega, \mathcal{P})$$

we get

$$\int \alpha - \epsilon \le S(f, \mathcal{P}) \le \int \omega + \epsilon.$$

The same is true for any other partition $\mathcal{Q}$ with $\|\mathcal{Q}\| < \delta$, s0

$$|S(f,\mathcal{P}) - S(f,\mathcal{Q})| < 2\epsilon.$$

Thus the Cauchy criterion holds. □

3. **Lemma 7.2.4:** If $J \subset I = [a,b]$ is a subinterval with endpoints $c < d$, and if $\varphi_J(x) = 1$ on J and 0 otherwise, then φ_J is integrable and $\int \varphi_J = d - c$.

Proof: Changing values at two points does not alter the integrability or integral so we may assume $J = [a,b]$. If we take a partition with norm δ, then the intervals outside J contribute zero and the intervals inside J contribute between $d-c$ and $d-c-2\delta$. There are at most 4 subintervals the hit the endpoints of J and these contribute between 0 and 4δ. Thus

$$|S(\varphi_J,\mathcal{P}) - (d-c)| \leq 6\delta,$$

which is $< \epsilon$ if $\delta < \epsilon/6$. □

4. **Theorem 7.2.5:** Any step function is Riemann integrable.

Proof: Every step function is a finite sum of functions as in the lemma, and we know finite sums of Riemann integrable functions is also integrable.

5. **Theorem 7.2.7:** Any continuous $f : [a,b] \to \mathbb{R}$ is Riemann integrable.

Proof: By Theorem 5.4.3, f is uniformly continuous. Given $\epsilon > 0$ choose δ so that $|x-y| < \delta$ implies $|f(x) - f(y)| < \epsilon/(b-a)$. Then for any tagged partition $\mathcal{P}$ define step functions α, ω that are constant on $[x_{k-1}mx_k)$ and equal to the min and max of f on the interval. Then

$$S(\omega,\mathcal{P}) - S(\ \alpha,\mathcal{P}) \leq \delta(b-a) < \epsilon.$$

Thus f is integrable by the Squeeze theorem. □

6. **Theorem 7.2.8:** Any monotone $f : [a,.b] \to \mathbb{R}$ is Riemann integrable.

Proof: Assume f is increasing (otherwise take $-f$). Partition $[a,b]$ and define α on I_k by $\alpha = f(x_{k-1})$ and define ω on I_k by $\omega = f(x_k)$. Then $\alpha \leq f \leq \omega$. Also

$$\int \alpha = \frac{b-a}{n}(f(x_0) + \cdots + f(x_{n-1}),$$

$$\int \omega = \frac{b-a}{n}(f(x_1) + \cdots + f(x_n),$$

so

$$\int \omega - \int \alpha = \frac{b-a}{n}(f(x_n) - f(x_0),$$

and this tends to zero at $n \nearrow \infty$. □

Finished here last time.

7. **7.2.9 Additivity Theorem:** Let $f : [a,b] \to \mathbb{R}$ and suppose $a < c < b$. Then $f \in \mathcal{R}[a,b]$ iff it restrictions are in $\mathcal{R}[a,c]$ and $\mathcal{R}[c,b]$, and $\int_a^b f = \int_a^c f + \int_c^b f$.

Proof:

($\Leftarrow$) First suppose the restriction are integrable. Given $\epsilon > 0$ choose $\delta > 0$ so that given tagged partitions of $[a, c]$ and $[c, b]$ the corresponding Riemann sums are within ϵ of the integrals $\int_a^c f$ and $\int_c^b f$. Since f is integrable on each interval, it is bounded on each and hence $|f|$ is bounded by some M on their union. By making δ smaller, we may assume $\delta < \epsilon/6M$.

Let $\mathcal{Q}$ be a tagged partition of $[a, b]$ with norm $< \delta$. We claim

$$S(f, \mathcal{Q}) - \int_a^c f - \int_c^b f| < \epsilon/3.$$

If c is a partition point of $\mathcal{Q}$ then this is easy, since the Riemann sum for $\mathcal{Q}$ splits into sums for the two subintervals.

Otherwise $c \in I_k$ for some k. We form partitions $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of $[a, c]$ and $[c, b]$ by adding c to the partition $\mathcal{Q}$. Then

$$|S(f, \mathcal{Q}) - S(f, \mathcal{Q}_1) - S(f, \mathcal{Q}_2)| \leq 2M\delta < \epsilon/3.$$

This implies

$$|S(f, \mathcal{Q}) - \int_a^c f \int_c^b f| \leq |\int_a^c f - S(f, \mathcal{Q}_1)| + |\int_c^b -S(f, \mathcal{Q}_2)| < \epsilon. \quad \square$$

8. **Corollary 7.2.10:** Restricting an integrable function to a subinterval gives an integrable function.
9. **Defn:** $\int_b^a f = -\int_a^b f$ and $\int_a^a f = 0$.
10. **Theorem 7.2.13:** If $f \in \nabla[a, b]$ and $\alpha, \beta, \gamma \in [a, b]$ then

$$\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\alpha f.$$

Proof: several cases to check. See text.

Section 7.3: The Fundamental Theorem

1. **Defn:** Suppose $f : [a, b] \to \mathbb{R}$ and $z \in [a, b]$ we define the indefinite integral

$$F(z) = \int_a^z f(x)dx$$

2. **Theorem 7.3.1 Fundamental Thm of Calculus 1st Form:** Suppose there is a finite set E in $[a, b]$ and functions f, F so that
   (a) F is continuous on $[a, b]$
   (b) $F'(x) = f(x)$ for all $x \in [a, b] \setminus E$.
   (c) $f \in \mathcal{R}[a, b]$

The we have $\int_a^b f = F(b) - F(a)$.
Proof: We preove this when $E = \{a, b\}$; the general case canbe obtained by breaking the interval into a finite number of subintervals.

Let $\epsilon > 0$. There is a $\delta > 0$ so that for any tagged partition with norm $< \delta$ the correspodning Riemann sum is within ϵ of $\int_a^b f$. The mean value theorm appplied to each subinterval implies

$$F(x_k) - F(x_{k-1}) = F'(u_k)(x_k - x_{k-1})$$

for some $u_k \in (x_k, x_{k-1})$. Thus

$$F(b) - F(a) = \sum_{k=1}^{n} F(x_k) - F(x_{k-1}) = \sum_{k=1}^{n} f(x_k) - f(x_{k-1}) = S(f, \mathcal{Q}),$$

for some parition $\mathcal{Q}$ with norm $< \delta$. Thus

$$|F(b) - F(a) - \int_a^b f| = |S(f, \mathcal{Q}) - \int_a^b f| < \epsilon. \quad \square$$

2. **Theorem 7.3.4:** The indefinite integral is continuous. In fact it is Lipschitz:

$$|F(z) - F(w)| \leq M|z - w|$$

where $M = \sup_I |f|$.
Proof:

$$|F(z) - F(w) = |\int_w^z f(x)dx| \leq M|z - w|. \quad \square$$

3. **Fundamental Theorem of Calculus:** Let $f \in \mathcal{R}[a, b]$ and let f be continuous at $z \in [1, b]$. Then the indefinite integral F of f is differentiable at c and $F'(c) = f(c)$.
Proof: Suppose $c \in [a, b)$ and consider the right hand derivative. Given $\epsilon > 0$ choose $\delta > 0$ so that for e $0 < h < \delta$ we have

$$f(c) - \epsilon < f(x) < f(c) + \epsilon$$

for $c < x < x + \delta$. By the Additivity theorem

$$F(c + h) - F(c) = \int_c^{c+h} f$$

$$(f(c) - \epsilon)h < F(c + h) - F(c) < (f(c) + \epsilon)h)$$

$$f(c) - \epsilon < \frac{F(c + h) - F(c)}{h} < f(c) + \epsilon$$

$$\epsilon < \frac{F(c + h) - F(c)}{h} - f(c) < \epsilon$$

Thus

$$\lim_{h \to 0+} \frac{F(c + h) - F(c)}{h} = f(c).$$

The proof for the left hand derivative is similar, and proves the theorem. $\square$

4. **Theorem 7.3.6:** If $f \in C([a, b])$ (f is continuous on $[a, b]$), then its indefinite integral F is differentiable on $[a, b]$ and $F' = f$.
5. Examples:
Step function

Thomae's function

6. **7.3.8 Substitution Theorem:** Let $J = [\alpha, \beta]$ and $\phi : J \to \mathbb{R}$ have a continuous derivative. If $f : I \to \mathbb{R}$ is continuous on an interval I containing $\varphi(J)$ then

$$\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx.$$

Proof: Exercise 17.

7. **Defn:** A set $Z \subset \mathbb{R}$ is a null set (or has zero length) if for any $\epsilon > 0$ there is a countable set of intervals $\{I_n\} = \{(a_n, b_n)\}$ so that

$$Z \subset \bigcup I_n \text{ and } \sum_n (b_n - a_n) < \epsilon.$$

Lemma: A subset of a null set is also a null set.

Lemma: A countable union of null sets is null.

8. $\mathbb{Q}$ is a null set. Any countable set is null.

9. Cantor's Middle third set. Give non-rigorous discussion.

10. **Defn:** A statement holds almost everywhere it is true for every real number in $\mathbb{R} \setminus Z$ where Z is a null set.

For example, almost every real number is irrational.

11. **Lebesgue's Integrability Criterion:** A bounded function $f : [a, b] :\to \mathbb{R}$ is Riemann integrable iff it is continuous almost everywhere on $[a, b]$.

12. **Composition theorem:** If $f : [a, b] \to [c, d]$ is Riemann integrable and φ is continuous on $[c, d]$ then $\varphi \circ f$ is Riemann integrable,

Proof: the composition is continuous everywhere f is, so the discontinuity set of $\varphi \circ f$ is a subset of the discontinuities of f. Hence it is also a null set, so the composition is integrable.

13. Corollary 7.3.15: If $f \in \mathcal{R}[a, b]$ so is $|f|$ and f^2.

14. **The Product Theorem:** If $f, g \in \mathcal{R}[a, b]$ then $fg \in \mathcal{R}[q, b]$.

Proof:

$$fg = \frac{1}{2}((f + g)^2 - f^2 - g^2),$$

and everything on the right is integrable since sums of integrable functions are integrable and f^2 is the composition of f with the continuous function x^2.

15. **Integration by Parts:** Let F, g be differentiable on $[a, b]$ and assume $f = F'$ and $g = G'$ are Riemann integrable on $[a, b]$. Then

$$\int_a^b fG = FG|_a^b - \int_a^b Fg.$$

Proof: By the product rule

$$(FG)' = F'G + FG' = fG + Fg$$

and functions on right are both integrable. So Fundamental theorem says

$$FG|_a^b = \int_a^b fG + \int_a^b fG. \quad \square$$

16. **Taylor's Theorem with the Remainder:** Suppose $f', \dots f^{(n+1)}$ exist of $[a,b]$ and that the last is Riemann integrable. Then

$$f(b) = f(a) + f(z) + f'(a)(b-a) + \cdots + \frac{f^{(n)}(a)}{n}(b-a)^n + R_n$$

where

$$R_n = \frac{1}{n}\int_a^b f^{(n+1)}(t)(b-t)^n dt.$$

Proof: Apply integration by parts to the remainder term with $F(t) = f^{(n)}(t)$ and $G(t) = (b-t)^n/n$ to get

$$\begin{aligned} R_n &= \frac{1}{n} f^{(n)}(t)(b-t)^n|_a^b + \frac{1}{(n-1)}\int_a^b f^{(n)}(t)(b-a)^{n-1}dt \\ &= -\frac{f^{(n)}(a)}{n}(b-a)^n + \frac{1}{(n-1)}\int_a^b f^{(n)}(t)(b-a)^{n-1}dt \end{aligned}$$

Continuing in this was we eventually reach the desired equation.

Appendix C: Riemann and Lebesgue Criteria

1. **Riemann Integrability Criterion:** Suppose $f : [a,b] \to \mathbb{R}$ is bounded. TFAE:
(a) f is Riemann integrable.
(b) For every $\epsilon > 0$ there is partition $\mathcal{P}$ so that if $\mathcal{P}_2$, $\mathcal{P}_2$ are tagged partitions with the same intervals as $\mathcal{P}$ then

$$|S(f,\mathcal{P}_1) - S(f,\mathcal{P}_1)| < \epsilon.$$

(c) For every $\epsilon > 0$ there is partition $\mathcal{P}$ so that if $m_k = \inf_{I_k} f$ and $M_k = \inf_{I_k} f$ then

$$\sum_k (M_k - m_k)(x_k - x_{k-1}) < 2\epsilon.$$

Proof:
(a) $\Rightarrow$ (b) is the Cauchy Criterion
(b) $\Rightarrow$ (c) On each interval I_k choose tags u_k, v_k so that

$$f(u_k) < m_k + \epsilon/(b-a), f(v_k) > M_k - \epsilon/(b-a)$$

Then for these tagged partitions

$$\begin{aligned} \sum_k (M_k - m_k)(x_k - x_{k-1}) &\le |S(f,\mathcal{P}_1) - S(f,\mathcal{P}_2)| + \sum (f(v_k) - f(u_k))\epsilon/(b-a) \\ &< \epsilon + \epsilon \end{aligned}$$

(c) $\Rightarrow$ (a) Define step functions $\alpha = m_k$ on I_k and $\omega = M_k$ on I_k. Then

$$\int \omega - \alpha = \sum_k (M_k - m_k)(x_k - x_{k-1}) < 2\epsilon$$

so f is integrable by the Squeeze theorem. $\square$

2. **Defn:** Suppose $f : A \to \mathbb{R}$ is bounded. For $S \subset A$ define the osscilation of f on S to be

$$W(f, S) = \sup_S f - \inf_S f.$$

3. **Defn:** An r-neighborhood of a point $c \in A$ is

$$V_r(c) = \{x \in A : |x - c| < r\}.$$

4. **Defn:** If $c \in A$, the oscillation of f at c is

$$w(f, c) = \inf\{W(f, V(r, c)) : r > 0\} = \inf_{r \to 0^+} W(f, V_r(c)).$$

Also if c is a cluster point of A, then

$$W(f, c) = \limsup_{x \to c} f(x) - \liminf_{x \to c} f(x).$$

If f has a jump discontinuity at c, this is the size of the jump.

From Section 5.5:

5. **Defn:** A gauge on $I = [a, b]$ is a positive function.

6. **Defn:** Given a gauge on $I = [a, b]$, a tagged partition is called δ-fine if for all k we have $I_k \subset V_{\delta(t_k)}(t_k)$.

7. **Theorem 5.5.5:** If δ is a gauge on $[a, b]$ then there exists a δ-fine partition.

Proof: Let E be the set of $x \in [a, b]$ so that there is a δ fine partition of $[a, x]$. Then E is not empty since $x \in E$ if $x < a + \delta(a)$ (we can use one interval for the partition).

Since $E \subset [a, b]$, $u = \sup E$ exists and $u \le b$. We want to prove $u \in E$ and $u = b$.

Choose $v \in E$ with $u - \delta(u) < v < u$. Then $[a, v]$ has a δ-fine partition and adding $[v, u]$ gives a δ-fine partition of $[a, u]$. Thus $u \in E$.

If $u < b$ then choose v so that $u < v < min(b, u + \delta(u))$ and add $[u, v]$ to a δ-fine partition of $[a, u]$ to get a δ-fine partition of $[a, v]$. This contradicts definition of supremum, so $u = b$. $\square$

8. **Lebesgue's Criterion:** A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable iff it is continuous almost everywhere on $[a, b]$.

Proof:

($\Rightarrow$: integrable implies a.e. continuous)

Let $H_n = \{x \in [a, b] : w(f, x) > 2^{-n}\}$. It is enough to show each H_k has measure zero. Thus for any ϵ it is enough to show H_n is covered by intervals with total length $< \epsilon$.

We use Part (3) of Riemann's criterion. Suppose we have a partition $\mathcal{P}$ so that

$$\sum_k (M_k - m_k)(x_k - x_{k-1}) < 2^{-n}\epsilon$$

If $x \in H_n \cap (x_k, x_{k-1})$ then $M_k - m_k \geq 2^{-n}$, so

$$2^{-n} \sum_k (x_k - x_{k-1}) < 2^{-n}\epsilon$$

$$\sum_k (x_k - x_{k-1}) < \epsilon$$

Since H_n is contained in these intervals, plus their finite set of endpoints, H_n has length $< \epsilon$ for all $\epsilon > 0$. Hence H_n is a null set, as desired.

($\Leftarrow$: a.e. continuous implies integrable)

Suppose $|f| \leq M$ on $[a, b]$ and assume the set D of discontinuities is a null set. Given $\epsilon > 0$ there is a covering of D by open intervals $\{J_k\}$ so that $\sum_k \ell(J_k) < \epsilon/4M$. Define a gauge on $[a, b]$ so that

(i) if $x \in D$ then $V_{\delta(x)}(x) \subset J_k$ for some k.

(ii) if $x \notin D$, then $y \in V_{\delta(x)}(x)$ implies $|f(x) - f(y)| < \epsilon/2(b-a)$.

By Theorem 5.5.5 there is a δ-fine partition $\mathcal{P}$ of $[a, b]$. Choose some tags $\{t_k\}$ for $\mathcal{P}$. Split the indices into two groups S_d and S_c depending on whether $t_k \in D$ or not. Then

$$\begin{aligned}
\sum_k (M_k - m_k)(x_k - x_{k-1}) &= \sum_{S_c} (M_k - m_k)(x_k - x_{k-1}) + \sum_{S_d} (M_k - m_k)(x_k - x_{k-1}) \\
&\leq \sum_{S_c} (\epsilon/2)(x_k - x_{k-1}) + \sum_{S_d} 2M(x_k - x_{k-1}) \\
&\leq (\epsilon/2(b-a))(b-a) + 2M(\epsilon/4M) \\
&= \epsilon.
\end{aligned}$$

Thus f is Riemann integrable by Part (3) of Riemann's criterion. $\square$