320 Fall 2021, Thursday, Nov 18, 2021

Section 9.1: Absolute Convergence

1. **Defn:** We say $\sum_n x_n$ is **absolutely convergent** if $\sum_n |x_n|$ converges. If $\sum_n x_n$ converges, but not absolutely, then it is **conditionally convergent**.
2. **Theorem 9.1.2:** If a series is absolutely convergent then it converges.
Proof: Use the triangle inequality and the Cauchy criterion:

$$|x_n + \dots x_m| \leq |x_n| + \dots |x_m|.$$

to show:
$\sum |x_n|$ converges $\Rightarrow \sum^m |x_n|$ Cauchy
$\Rightarrow \sum^m x_n$ Cauchy
$\Rightarrow \sum^m x_n$ converges.
3. **Theorem 9.1.3:** if $\sum x_n$ is convergent, then any series obtained by grouping terms is also convergent and to the same value.
Proof: The partial sums of the grouped series form a subsequence of the partial sums of the un-grouped series, and hence converge to the same limit.
4. Same does not hold for un-grouping, e.g. $(-1+1_+(-1+1)+\dots$.
5. **Defn:** $\sum y_n$ is a **rearrangement** of $\sum x_n$ if $y_k = x_{f(k)}$ for some bijection $f : \mathbb{N} \to \mathbb{N}$.
6. **Rearrangement Theorem:** If $\sum x_n$ is absolutely convergent then any rearrangement also converges to the same limit.
Proof: Suppose $\sum x_n = x$. If $\epsilon > 0$ choose N so that

$$|x - s_n| < \epsilon \text{ and } \sum_N^n x_k < \epsilon,$$

for all $n > N$. Choose M so that all the terms $x_1, \dots x_N$ are contained in $y_1, \dots y_M$. If $m > M$ then the difference of partial sums $s_N - s_m$ is a finite sum of terms x_k that come after N and so is bounded by ϵ. Therefore

$$|s_m - x| \leq |x - s_N| + |s_N - s_m| \leq 2\epsilon. \quad \square$$

Section 9.2: Tests for Absolute Convergence

1. **Limit Comparison Test:** If (x_n), (y_n) are non-zero sequences

$$\lim_n \frac{x_n}{y_n} = r$$

exists, then
(a) if $0 < r < \infty$ then $\sum x_n$ is absolutely convergent iff $\sum y_n$ is.

(b) if $r = 0$ and $\sum y_n$ is absolutely convergent then $\sum x_n$ is.

2. **Root test:** Let $(x_n$ be a sequence in $\mathbb{R}$.
(a) if there is $r \in [0, 1)$ and $K \in \mathbb{N}$ so that $n \geq K$ implies

$$|x_n|1/n \leq r,$$

then $\sum x_n$ is absolutely convergent.
(b) if there is a $K \in \mathbb{N}$ so that $n \geq K$ implies

$$|x_n|1/n \leq 1,$$

then $\sum x_n$ is divergence..

3. **Corollary 9.2.3:** If $r = \lim x_n^{1/n}$ exists then $\sum x_n$ is convergent if $r < 1$ and divergent if $r > 1$.

4. **Ratio Test:**
(a) if there is an $r \in (0, 1)$ and a $K \in \mathbb{N}$ so that $n \geq K$ implies

$$\frac{|x_{n+1}|}{|x_n|} \leq r,$$

then $\sum x_n$ is absolutely convergent.
(b) if there is a $K \in \mathbb{N}$ so that $n \geq K$ implies

$$\frac{|x_{n+1}|}{|x_n|} \leq 1,$$

then $\sum x_n$ is divergent.

5. **Corollary 9.2.5:** If $r = \lim |x_{n+1}|/|x_n|$ exists then $\sum x_n$ is convergent if $r < 1$ and divergent if $r > 1$.

6. **The Integer-Al Test:** Let f be a positive, decreasing function on $[1, \infty)$. Then the series $s = \sum_k f(k)$ converges iff the improper integral

$$\int_1^\infty f(x)dx = \lim_{b \to \infty} \int_1^b f(x)dx$$

exists. If if converges, the partial sums of the series satisfy

$$\int_{n+1}^\infty f(x)dx \leq s - s_n \leq \int_n^\infty f(x)dx.$$

7. Examples:
$\sum n^{-2}$.
$\sum \frac{1}{n \log^2 n}$

8. **Raabe's Test:** Let (x_n) be a sequence of non-zero reals.
(a) if there are $a > 1$ and $K \in \mathbb{N}$ so that $n > K$ implies

$$\left| \frac{x_{n+1}}{x_n} \right| \leq 1 - \frac{a}{n}.$$

then $\sum x_n$ is absolutely convergent.

(b) if there are $a \leq$ and $K \in \mathbb{N}$ so that $n > K$ implies

$$\left|\frac{x_{n+1}}{x_n}\right| \geq 1 - \frac{a}{n}.$$

then $\sum x_n$ is absolutely convergent.

Proof:

(a) Rearranging, We get

$$(k-1)|x_k) - k|x_{k+1}| \geq (a-1)|x_k|.$$

for $kgeqK$. Hence $k|x_{k+1}|$ is decreasing. If we note the left sides telescopes,

$$(K-1)|K_k) - n|x_{n+1}| \geq (a-1)(|x_K| + \cdots + |x_n|).$$

This shows the partial sums are bounded, and hence the series $\sum x_n$ converges absolutely.

(b) The same reasoning shows $k|x_{k+1}|$ is eventually increasing and hence x_n is bounded below by $C/(n-)$. Since the harmonic series diverges, so does (x_n).

10. **Corollary:** Assume

$$a = \lim(n(1 - \frac{|x_{n+1}|}{|x_n|}$$

exists. If $a > 1$ then $\sum x_n$ is absolutely convergent, and if $a < 1$ is it is not absolutely convergent.

Proof: Suppose the limit exists and is > 1. If $1 < a_1 < a$ then $a_1 < n(1 - |x_{n+1}/x_n|)$ for sufficiently large n, so $|x_{n+1}/x_n| < 1 - a_1/n$ and Raabe's test applies.

If $a < 1$, the argument is similar.

Section 9.3: Test for Non-absolute Convergence

1. **Defn:** A non-zero series (x_n) is **alternating** if the x_{n+1} has the opposite sign to x_n.

2. **Alternating Series Test:** If (z_n) decreases to 0 then the alternating series $\sum(-1)^n z_n$ converges.

Proof: Note that

$$s_{2n} = (z_1 - z_2) + \ldots (z_{2n-1} - z_{2n}),$$

is increasing and bounded above by

$$s_{2n} = z_1 - (z_2 - z_3) + \cdots - z_{2n}) \leq z_1$$

Hence the even partial sums converge to a limit L by the Monotone convergence theorem.

A similar argument show the odd partial sums are decreasing and bounded, so converge to some M. But

$$|s_{2n} - s_{2n+1}| = |z_{2n+1}| \to 0$$

so the two sequence have the same limit, hence the whole sequence converges.

3. **Abel's lemma:** Let (x_n), (y_n) be sequences in $\mathbb{R}$ and let $s_n = \sum_{k=1}^n y_k$, with $s_0 = 0$. If $m > n$, then

$$\sum_{k=n+1}^{m} x_k y_k = (x_n s_n - x_{n+1} s_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) s_k.$$

Proof: Since $y_k = s_k - s_{k-1}$, the left side is If $m > n$, then

$$\sum_{k=n+1}^{m} x_k (s_k - s_{k-1}) = (x_{n+1} s_{n+1} - x_{n+1} s_n) + (x_{n+2} s_{n+2} - x_{n+2} s_{n+1}) + \ldots (x_m s_m - x_m s_{m-1}),$$

which is the right hand side.
4. **Dirichlet's Test:** If (x_n) decreases to 0 and if the partial sums (s_n) of $\sum y_n$ are bounded, then $\sum s_n y_n$ is convergent.
Proof: Suppose $|s_n| \leq B$. Since $x_k - x_{k+1} \geq 0$,

$$\begin{aligned} |\sum_{k=n+1}^{m} x_k y_k| &\leq (x_m + x_{n+1}) B + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) B \\ &\leq (x_m + x_{n+1}) + (x_{n+1} - x_B) B \\ &= 2 x_{n+1} B. \end{aligned}$$

Since $x_n \to 0$, the series satisfies the Cauchy criterion and converges.
Special case is alternating series $y_n = (-1)^n$.
5.

$$\sin\alpha \cos\beta = \frac{1}{2}(\sin(\alpha+\beta) - \sin(\beta-\alpha)).$$

Hence

$$2\sin(\frac{1}{2}x)(\cos kx) = \sin(k+\frac{1}{2})x - \sin(k-\frac{1}{2})x,$$

$$2\sin(\frac{1}{2}x)(\cos x + \cdots + \cos nx) = \sin(n+\frac{1}{2})x - \sin\frac{1}{2}x,$$

implies

$$|\cos x + \cdots + \cos nx| \leq \frac{1}{\sin(x/2)},$$

so if a_n decreases to 0,

$$\sum a_n \cos(nx)$$

converges for $x \neq 2\pi n$.
Similar result for $\sum a_n \sin nx$.

Section 9.4: Series of functions.

1. **Defn:** If (f_n) is a sequence of functions, let (s_n) denote the sequence of partial sums
$$s_n(x) = f_1(x) + \cdots + f_n(x).$$
If the partial sums converge (pointwise) to f we say $\sum f_n$ converges to f.
2. Can define absolutely convergent or uniformly convergent in obvious way.
3. **Theorem 9.4.2:** If (f_n) are all continuous and converge uniformly to f, then f is continuous.
3. **Theorem 9.4.3:** If (f_n) are all Riemann integrable on $J = [a, b]$ and $\sum f_n$ converges uniformly to f, then
$$\int f = \sum \int f_n.$$
3. **Theorem 9.4.4:** If (f_n) are all differentiable on $J = [a, b]$, that $\sum f_n$ converges at some point of J and that $\sum f_n'$ converges uniformly on J. Then there is a f so that $f - \sum f_n$ and $f' = \sum f_n'$.
4. **Theorem 9.4.5:** If (f_n) are functions $D \to \mathbb{R}$. The series $\sum f_n$ is uniformly convergent iff for all $\epsilon > 0$ there is a M so that $n > M$ implies
$$|f_{n+1}(x) + \dots f_m(x)| < \epsilon,$$
for all $x \in D$.
5. **Weierstrass M-test:** Let M_n be real numbers so that $|f_n(x)| \le M_n$ for all $x \in D$. If $\sum M_n$ converges, then $\sum f_n$ is uniformly convergent.
6. **Defn:** a power series around $c \in \mathbb{R}$ is a series of the form
$$\sum_{n=0}^{\infty} a_n (x - c)^n.$$
Most common case is $c = 0$.
6. A power series may converge only at c, on an interval $(c - r, c + r)$ or on all on $\mathbb{R}$.The limit of a convergent power series is called analytic.
8. Define
$$\rho = \limsup |a_n|^{1/n}.$$
The radius of convergence is $R = 0$ if $\rho = \infty$, $1/\rho$ if $0 < \rho < \infty$ and $R = \infty$ if $\rho = 0$.
9. **Cauchy-Hadamard:** If R is the radius of convergence of a power series $\sum a_n x^n$, then the series is absolutely convergent for $|x| < R$ and divergent for $|x| > R$.
10. All behaviors possible on boundary
$$\sum x^n, \sum \frac{1}{n} x^2, \sum \frac{1}{n^2} x^n.$$
11. **Theorem 9.4.10:** If $\sum a_n x^n$ has radius on convergence R then it converges on any closed, bounded interval $K \subset (-R, R)$.

12. **Theorem 9.4.11:** The limit of a power series is continuous on its interval of convergence. On any closed bounded subinterval, it can be integrated term-by-term
.
13. **Theorem 9.4.12:** The limit of a power series is differentiable on its open interval of convergence. If $f(x) = \sum a_n x_n$ then

$$f'(x) = \sum n a_n x^{n-1}.$$

and both functions have the same radius of convergence.
14. **Theorem 9.4.13:** If $\sum a_n x_n$ and $\sum b_n x_n$ converge to the same function on the same interval $(-r, r)$, then $a_n = b_n$ for all n.
Proof: by above $a_n = f^{(n)}(0)/n = b_n$. $\square$
15. If f has infinitely many derivatives, its Taylor series is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n}(x-c)^n.$$

If a power series converges to f it is the Taylor series of f.

But, just because a function is infinitely differentiable, its Taylor series need not converge to it. If the Taylor series of f does converge to f, then f is called analytic.

It is possible the Taylor series converges to the "wrong" function.
16. Examples:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} x^{2n+1}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)} x^{2n}$$

$$\exp x = \sum_{n=0}^{\infty} \frac{1}{n} x^n$$