320 Fall 2021, Tuesday, Nov 16, 2021

Section 8.1: Pointwise and Uniform Convergence

1. **Defn:** A sequence of functions (f_n) is a choice of function $f_n : A \to \mathbb{R}$ for each $n \in \mathbb{Z}$. The domain A should be the same for every n.
2. We say f_n converges pointwise to f on A if $f_n(x) \to f(x)$ for every $x \in A$.
3. Examples:
   $f_n(x) = x/n$ on $\mathbb{R}$
   $f_n(x) = x^n$ on $[0, 1]$.
4. **Defn:** We say f_n converges to f uniformly on A if for all $\epsilon > 0$ there is a K so that $n \geq K$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \in A$.
5. Uniform convergence implies pointwise convergence, but not conversely. We say uniform convergence is "stronger" than pointwise convergence.
6. **Lemma 8.1.5:** f_n does not converge uniformly on A to f iff there is some $\epsilon_0 > 0$ and a subsequence f_{n_k} and a sequence (x_k) so that

$$|f_{n_k}(x_k) - f(x_k)| \geq \epsilon_0.$$

7. **Defn:** We say $\varphi : A \to \mathbb{R}$ is bounded if $\varphi(A)$ is a bounded set, i.e., $|\varphi(x)| \leq M$ for some M and all $x \in A$. Define

$$\|\varphi\|_A = \sup\{|\varphi(x)| : x \in A\}.$$

8. Bounded functions on a set form a vector space. This is a norm on that vector space.
9. **Lemma 8.1.8:** A sequence f_n converges uniformly to f on A iff

$$\|f_n - f\|_A \to 0.$$

Proof: If $f_n \to f$ uniformly then for all ϵ there is a K so that $n \geq K$ implies

$$|f_n - f\|_A = \sup\{|f_n(x) - f(x) : x \in A\} \to 0.$$

Conversely, if $|f_n - f\|_A \to 0$, then for all ϵ there is a K so that $n \geq K$ implies $|f_n - f\|_A < \epsilon$, which is the same as $|f_n(x) - f(x)| < \epsilon$ for all $x \in A$.

10. **Cauchy Criterion for Uniform Convergence:** Suppose f_n is sequence of bounded functions on A. Then f_n converges uniformly on A to a bounded function f iff for all $\epsilon > 0$ there is a H so that for all $n, m \geq H$ we have $\|f_n - f_m\|_A < \epsilon$.

Proof: If $f_n \to$ uniformly then for any $\epsilon > 0$ there is a K so that $n \geq K$ implies $\|f - f_n\|_A < \epsilon/2$, so $n, m \geq K$ implies

$$\|f_m - f_n\|_A \|f - f_n\|_A + \|f - f_m\|_A < \epsilon/2 + \epsilon/2 < \epsilon.$$

Conversely, if the Cauchy condition holds for f_n, then for each $x \in A$ we have

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_A,$$

so $(f_n(x))$ is a Cauchy sequence of real numbers and hence converge so some limit we call $f(x)$. Since

$$|f(x) - f_m(x)| \leq \limsup_n |f_n(x) - f_m(x)| \leq \epsilon$$

if $m \geq H$, we see that $f_m \to f$ uniformly.

Section 8.2: Interchange of Limits

1. Questions:
Is a limit of continuous functions continuous? x^n
Is a limit of differentiable functions differentiable?
Is a limit of Riemann integrable functions Riemann integrable? Sliding tent.

2. **Theorem 8.2.2:** If (f_n) is a sequence of continuous function converging uniformly on A to f, then f is continuous on A.

Proof: Given $\epsilon > 0$ there is a H so that $n \geq H$ implies $|f_n(x) - f(x)| < \epsilon/3$ for all $x \in A$. Also, given $c \in A$ there exists $\delta > 0$ so that $|x - c| < \delta$ implies $|f_n(x) - f_n(c)| < \epsilon/3$. Thus for $|x - c| < \delta$, we have

$$|f(x) - f(c)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)| \leq \epsilon.$$

3. Partial sums of $\sum_{n=1}^{\infty} 2^{-n} \cos(3^n x)$ show that differentiable functions can converge uniformly to a nowhere differentiable function.

4. **Theorem 8.2.3:** Let $J = [a, b] \subset \mathbb{R}$ be a bounded interval and (f_n) a sequence of functions on J. Suppose there is a $x_0 \in J$ so that $f_n(x_0)$ converges and that f_n' converge uniformly to g on J. Then f_n converges uniformly to a a differentiable function f on J. so that $f' = g$.

Proof: Take some $x \in J$. Apply the mean value theorem to $f_n - f_m$ to find a point y between x_0 and x so that

$$f_m(x) - f_n(x) = f_m(x_0) - f_n(x_0) + (x - x_0)(f_m'(y) - f_n'(y)).$$

Thus

$$\|f_m(x) - f_n(x)\|_J \leq |f_m(x_0) - f_n(x_0)| + |b - a| \cdot \|f_m'(y) - f_n'()\|_J.$$

Hence $\{f_n\}$ is Cauchy and therefore convergent. Thus it has a continuous limit f.

Take $c \in J$. To prove $f'(c)$ exists, apply the mean value theorem between x and c to find a z between them so that

$$f_m(x) - f_n(x) - (f_m(c) - f_n(c)) = (x - c)(f_m'(z) - f_n'(z)).$$

$$|\frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c}| \leq \|f_m'(z) - f_n'(z)\|_J.$$

For any $\epsilon > 0$ there is an H so that $n, m \geq H$ imply

$$|\frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c}| \leq \epsilon.$$

Take the limit over m

$$|\frac{f(x)-f(c)}{x-c}-\frac{f_n(x)-f_n(c)}{x-c}|\leq \epsilon.$$

Since $g(c)=\lim_n f_n'(c)$, there is an N so that $n\geq N$ implies

$$|g(c)-f_n'(c)|<\epsilon.$$

Let $K=\max(H,N)$. Since $f_K'(c)$ exists, there is a $\delta>0$ so that $0<|x-c|<\delta$ implies

$$|\frac{f_K(x)-f_K(c)}{x-c)}-f_K'(c)<\epsilon.$$

Hence if $0<|x-c|<\delta$, we have

$$|\frac{f(x)-f(c)}{x-c}-g(c)|<3\epsilon.$$

Hence $f'(c)=g(c)$. □

5. **Theorem 8.2.4:** If (f_n) are Riemann integrable functions converging uniformly to f then f is Riemann integrable and

$$\int_a^b f=\lim_n \int_a^b f_n.$$

Proof: Given any $\epsilon>0$ there is an N so that $n>N$ implies

$$\alpha(x)=f_n(x)-\epsilon\leq f(x)\leq \omega(x)=f_n(x)+\epsilon$$

Both α and ω are Riemann integrable and

$$\int_a^n \omega-\alpha=\epsilon(b-a).$$

Thus by the squeeze theorem f is Riemann integrable and for all n

$$\int_a^b f_n(x)-\epsilon(b-a)\leq \int_a^b f(x)\leq \int_a^b f_n(x)+\epsilon(b-a)$$

$$|\int_a^b f-\int_a^b f_n|<\epsilon.$$

Thus $\int f_n\to\int f$. □

Both α and ω are Riemann integrable and

6. **Theorem 8.2.5:** Suppose (f_n) are Riemann integrable functions on $[a,b]$ converging pointwise to a to a Riemann integrable function f. Suppose also that there exists a B so that $|f_n(x)|\leq B$ for all n and all $x\in[a,b]$. Then $\int f_n\to\int f$.

Proof: see link on class webpage.

6. **Dini's Theorem:** Suppose (f_n) is a monotone sequence of continuous function on $I=[a,b]$ that converges pointwise to f. Then $f_n\to f$ uniformly.

Proof: We assume $f_1\geq f_2\geq\dots$. Let $g_n=f_n-f\geq 0$. It is enough to show $g_n\to 0$ uniformly.

Given $\epsilon > 0$ and $t \in I$ there is a M so that $0 \le g_m(t) < \epsilon$ for $m \ge M$. Since g_m is continuous there is a $\delta(t) > 0$ so that $|x - t| < \delta(t)$ implies $|g_m(t) - g_m(x)| < \epsilon$. Take $\delta(t)$ as a guage on I and let $\mathcal{P}$ be a δ-fine tagged partition. Let $M = \max(m_{t_1}, \ldots m_{t_n})$. If $m > M$ and $x \in I$ then there is an index k with $|x - t_k| < \delta(t_k)$ so

$$0 \le g_m(x) \le g_{m_k}(x) < \epsilon.$$

Thus $g_n \to 0$ uniformly. □

Planning to skip Sections 8.3 and 8.4 in text.

Section 9.1: Absolute Convergence

1. **Defn:** We say $\sum_n x_n$ is **absolutely convergent** if $\sum_n |x_n|$ converges. If $\sum_n x_n$ converges, but not absolutely, then it is **conditionally convergent**.
2. **Theorem 9.1.2:** If a series is absolutely convergent then it converges.
Proof: Use the triangle inequality and the Cauchy criterion:

$$|x_n + \ldots x_m| \le |x_n| + \ldots |x_m|.$$

to show:
$\sum |x_n|$ converges $\Rightarrow \sum^m |x_n|$ Cauchy
$\Rightarrow \sum^m x_n$ Cauchy
$\Rightarrow \sum^m x_n$ converges.
3. **Theorem 9.1.3:** if $\sum x_n$ is convergent, then any series obtained by grouping terms is also convergent and to the same value.
Proof: The partial sums of the grouped series form a subsequence of the partial sums of the un-grouped series, and hence converge to the same limit.
4. Same does not hold for un-grouping, e.g. $(-1 + 1_+(-1 + 1) + \ldots$.
5. **Defn:** $\sum y_n$ is a **rearrangement** of $\sum x_n$ if $y_k = x_{f(k)}$ for some bijection $f : \mathbb{N} \to \mathbb{N}$.
6. **Rearrangement Theorem:** If $\sum x_n$ is absolutely convergent then any rearrangement also converges to the same limit.
Proof: Suppose $\sum x_n = x$. If $\epsilon > 0$ choose N so that

$$|x - s_n| < \epsilon \text{ and } \sum_N^n x_k < \epsilon,$$

for all $n > N$. Choose M so that all the terms $x_1, \ldots x_N$ are contained in $y_1, \ldots y_M$. If $m > M$ then the difference of partial sums $s_N - s_m$ is a finite sum of terms x_k that come after N and so is bounded by ϵ. Therefore

$$|s_m - x| \le |x - s_N| + |s_N - s_m| \le 2\epsilon. \quad \square$$

Section 9.2: Tests for Absolute Convergence

1. **Limit Comparison Test:** If (x_n), (y_n) are non-zero sequences

$$\lim_n \frac{x_n}{y_n} = r$$

exists, then

(a) if $0 < r < \infty$ then $\sum x_n$ is absolutely convergent iff $\sum y_n$ is.

(b) if $r = 0$ and $\sum y_n$ is absolutely convergent then $\sum x_n$ is.

2. **Root test:** Let $(x_n$ be a sequence in $\mathbb{R}$.

(a) if there is $r \in [0, 1)$ and $K \in \mathbb{N}$ so that $n \geq K$ implies

$$|x_n|1/n \leq r,$$

then $\sum x_n$ is absolutely convergent.

(b) if there is a $K \in \mathbb{N}$ so that $n \geq K$ implies

$$|x_n|1/n \leq 1,$$

then $\sum x_n$ is divergence..

3. **Corollary 9.2.3:** If $r = \lim x_n^{1/n}$ exists then $\sum x_n$ is convergent if $r < 1$ and divergent if $r > 1$.

4. **Ratio Test:**

(a) if there is an $r \in (0, 1)$ and a $K \in \mathbb{N}$ so that $n \geq K$ implies

$$\frac{|x_{n+1}|}{|x_n|} \leq r,$$

then $\sum x_n$ is absolutely convergent.

(b) if there is a $K \in \mathbb{N}$ so that $n \geq K$ implies

$$\frac{|x_{n+1}|}{|x_n|} \leq 1,$$

then $\sum x_n$ is divergent.

5. **Corollary 9.2.5:** If $r = \lim |x_{n+1}|/|x_n|$ exists then $\sum x_n$ is convergent if $r < 1$ and divergent if $r > 1$.

6. **The Integer-Al Test:** Let f be a positive, decreasing function on $[1, \infty)$. Then the series $s = \sum_k f(k)$ converges iff the improper integral

$$\int_1^\infty f(x)dx = \lim_{b\to\infty} \int_1^b f(x)dx$$

exists. If if converges, the partial sums of the series satisfy

$$\int_{n+1}^\infty f(x)dx \leq s - s_n \leq \int_n^\infty f(x)dx.$$

7. Examples:

$\sum n^{-2}$.

$\sum \frac{1}{n \log^2 n}$

8. **Raabe's Test:** Let (x_n) be a sequence of non-zero reals.

(a) if there are $a > 1$ and $K \in \mathbb{N}$ so that $n > K$ implies

$$\left|\frac{x_{n+1}}{x_n}\right| \leq 1 - \frac{a}{n}.$$

then $\sum x_n$ is absolutely convergent.

(b) if there are $a \leq$ and $K \in \mathbb{N}$ so that $n > K$ implies

$$\left|\frac{x_{n+1}}{x_n}\right| \geq 1 - \frac{a}{n}.$$

then $\sum x_n$ is absolutely convergent.

Proof:

(a) Rearranging, We get

$$(k-1)|x_k) - k|x_{k+1}| \geq (a-1)|x_k|.$$

for $kgeqK$. Hence $k|x_{k+1}|$ is decreasing. If we note the left sides telescopes,

$$(K-1)|K_k) - n|x_{n+1}| \geq (a-1)(|x_K| + \cdots + |x_n|).$$

This shows the partial sums are bounded, and hence the series $\sum x_n$ converges absolutely.

(b) The same reasoning shows $k|x_{k+1}|$ is eventually increasing and hence x_n is bounded below by $C/(n-)$. Since the harmonic series diverges, so does (x_n).

10. **Corollary:** Assume

$$a = \lim(n(1 - \frac{|x_{n+1}|}{|x_n|}$$

exists. If $a > 1$ then $\sum x_n$ is absolutely convergent, and if $a < 1$ is it is not absolutely convergent.

Proof: Suppose the limit exists and is > 1. If $1 < a_1 < a$ then $a_1 < n(1 - |x_{n+1}/x_n|)$ for sufficiently large n, so $|x_{n+1}/x_n| < 1 - a_1/n$ and Raabe's test applies.

If $a < 1$, the argument is similar.

Section 9.3: Test for Non-absolute Convergence

1. **Defn:** A non-zero series (x_n) is **alternating** if the x_{n+1} has the opposite sign to x_n.

2. **Alternating Series Test:** If (z_n) decreases to 0 then the alternating series $\sum(-1)^n z_n$ converges.

Proof: Note that

$$s_{2n} = (z_1 - z_2) + \ldots (z_{2n-1} - z_{2n}),$$

is increasing and bounded above by

$$s_{2n} = z_1 - (z_2 - z_3) + \cdots - z_{2n}) \leq z_1$$

Hence the even partial sums converge to a limit L by the Monotone convergence theorem.

A similar argument show the odd partial sums are decreasing and bounded, so converge to some M. But

$$|s_{2n} - s_{2n+1}| = |z_{2n+1}| \to 0$$

so the two sequence have the same limit, hence the whole sequence converges.

3. **Abel's lemma:** Let (x_n), (y_n) be sequences in $\mathbb{R}$ and let $s_n = \sum_{k=1}^{n} y_k$, with $s_0 = 0$. If $m > n$, then

$$\sum_{k=n+1}^{m} x_k y_k = (x_n s_n - x_{n+1} s_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) s_k.$$

Proof: Since $y_k = s_k - s_{k-1}$, the left side is If $m > n$, then

$$\sum_{k=n+1}^{m} x_k(s_k - s_{k-1}) = (x_{n+1}s_{n+1} - x_{n+1}s_n) + (x_{n+2}s_{n+2} - x_{n+2}s_{n+1}) + \ldots (x_m s_m - x_m s_{m-1}),$$

which is the right hand side.

4. **Dirichlet's Test:** If (x_n) decreases to 0 and if the partial sums (s_n) of $\sum y_n$ are bounded, then $\sum s_n y_n$ is convergent.

Proof: Suppose $|s_n| \leq B$. Since $x_k - x_{k+1} \geq 0$,

$$\begin{aligned} |\sum_{k=n+1}^{m} x_k y_k| &\leq (x_m + x_{n+1})B + \sum_{k=n+1}^{m-1} (x_k - x_{k+1})B \\ &\leq (x_m + x_{n+1}) + (x_{n+1} - x_B)B \\ &= 2x_{n+1}B. \end{aligned}$$

Since $x_n \to 0$, the series satisfies the Cauchy criterion and converges.

Special case is alternating series $y_n = (-1)^n$.

5.

$$\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha+\beta) - \sin(\beta-\alpha)).$$

Hence

$$2\sin(\frac{1}{2}x)(\cos kx) = \sin(k+\frac{1}{2})x - \sin(k-\frac{1}{2})x,$$

$$2\sin(\frac{1}{2}x)(\cos x + \cdots + \cos nx) = \sin(n+\frac{1}{2})x - \sin\frac{1}{2}x,$$

implies

$$|\cos x + \cdots + \cos nx| \leq \frac{1}{\sin(x/2)},$$

so if a_n decreases to 0,

$$\sum a_n \cos(nx)$$

converges for $x \neq 2\pi n$.

Similar result for $\sum a_n \sin nx$.

Section 9.4: Series of functions.

1. **Defn:** If (f_n) is a sequence of functions, let (s_n) denote the sequence of partial sums

$$s_n(x) = f_1(x) + \cdots + f_n(x).$$

If the partial sums converge (pointwise) to f we say $\sum f_n$ converges to f.
2. Can define absolutely convergent or uniformly convergent in obvious way.
3. **Theorem 9.4.2:** If (f_n) are all continuous and converge uniformly to f, then f is continuous.
3. **Theorem 9.4.3:** If (f_n) are all Riemann integrable on $J = [a, b]$ and $\sum f_n$ converges uniformly to f, then

$$\int f = \sum \int f_n.$$

3. **Theorem 9.4.4:** If (f_n) are all differentiable on $J = [a, b]$, that $\sum f_n$ converges at some point of J and that $\sum f'_n$ converges uniformly on J. Then there is a f so that $f - \sum f_n$ and $f' = \sum f'_n$.
4. **Theorem 9.4.5:** If (f_n) are functions $D \to \mathbb{R}$. The series $\sum f_n$ is uniformly convergent iff for all $\epsilon > 0$ there is a M so that $n > M$ implies

$$|f_{n+1}(x) + \dots f_m(x)| < \epsilon,$$

for all $x \in D$.
5. **Weierstrass M-test:** Let M_n be real numbers so that $|f_n(x)| \leq M_n$ for all $x \in D$. If $\sum M_n$ converges, then $\sum f_n$ is uniformly convergent.
6. **Defn:** a power series around $c \in \mathbb{R}$ is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n.$$

Most common case is $c = 0$.
6. A power series may converge only at c, on an interval $(c - r, c + r)$ or on all on $\mathbb{R}$.The limit of a convergent power series is called analytic.
8. Define

$$\rho = \limsup |a_n|^{1/n}.$$

The radius of convergence is $R = 0$ if $\rho = \infty$, $1/\rho$ if $0 < \rho < \infty$ and $R = \infty$ if $\rho = 0$.
9. **Cauchy-Hadamard:** If R is the radius of convergence of a power series $\sum a_n x^n$, then the series is absolutely convergent for $|x| < R$ and divergent for $|x| > R$.
10. All behaviors possible on boundary

$$\sum x^n, \sum \frac{1}{n} x^2, \sum \frac{1}{n^2} x^n.$$

11. **Theorem 9.4.10:** If $\sum a_n x^n$ has radius on convergence R then it converges on any closed, bounded interval $K \subset (-R, R)$.
12. **Theorem 9.4.11:** The limit of a power series is continuous on its interval of convergence. On any closed bounded subinterval, it can be integrated term-by-term
.
13. **Theorem 9.4.12:** The limit of a power series is differentiable on its open interval of convergence. If $f(x) = \sum a_n x_n$ then

$$f'(x) = \sum n a_n x^{n-1}.$$

and both functions have the same radius of convergence.
14. **Theorem 9.4.13:** If $\sum a_n x_n$ and $\sum b_n x_n$ converge to the same function on the same interval $(-r, r)$, then $a_n = b_n$ for all n.
Proof: by above $a_n = f^{(n)}(0)/n = b_n$. □
15. If f has infinitely many derivatives, its Taylor series is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n}(x-c)^n.$$

If a power series converges to f it is the Taylor series of f.

But, just because a function is infinitely differentiable, its Taylor series need not converge to it. If the Taylor series of f does converge to f, then f is called analytic.

It is possible the Taylor series converges to the "wrong" function.
16. Examples:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} x^{2n+1}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)} x^{2n}$$

$$\exp x = \sum_{n=0}^{\infty} \frac{1}{n} x^n$$