**MAT 319 & 320 Fall 2021, Lecture 1, Tuesday Aug 24, 2021**

1. Introduce self, Prof Martens.
2. Explain joint nature of MAT 319/320. Alternate weeks teaching.
3. Webpages.
4. Textbook "Introduction to Real Analysis" 4th edition by Bartle and Sherbert.
5. First exam Thursday, Sept 30, 2021. Split class based on results. MAT 319 stays in this room, MAT 320 moves to P-131 in Math Tower.
6. Problem sets due Monday in recitations. TAs will set policy for how, when to hand in. Goal is to grade and return following Monday. Will discuss solutions after problem sets are handed in.
7. Office hours.
8. Questions about administration?
9. Pre-requisites: Chapter I.
   - Section 1.1: sets and functions
   - Section 1.2: induction
   - Section 1.3: finite and infinite sets

10. Set notation
   $A, B$ are sets.
   $A \subset B$, $A \supset B$
   Venn diagram
   $a \in A$
   $A \cup B$ union
   $A \cap B$ intersection
   $A \setminus B$ complement, set difference
   $A \Delta B = (A \setminus B) \cup (B \setminus A)$ symmetric difference
   Prove: $A \Delta B = (A \cup B) \setminus (A \cap B)$.
   $A \times B$ Cartesian product
11. $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R} \setminus \mathbb{Q}$, $\mathbb{Z} \setminus 2 \cdot \mathbb{Z}$
12. A function $f : A \to B$ is a subset of $A \times B$ so that $(a, b) \in f$ and $(a, c) \in f$ implies $b = c$.
13. Vertical line test.
13. exotic examples: $f(x) = 0$ if $x$ is irrational, $f(x) = 0$ if $x$ if rational.
14. Direct image: $f(E) = \{f(x) : x \in E\}$.
15. What is direct image of $\mathbb{R}$ under $f(x) = x^2$?
15. What is direct image of $\mathbb{Z}$ under $f(x) = x^2$?
16. $f$ is injective (1-to-1) if $f(x) = f(y)$ implies $x = y$.
17. $f$ is surjective from $A$ to $B$ if $f(A) = B$, i.e., for all $b \in B$ there is an $a \in A$ with $f(a) = b$.
18. bijective = injective + surjective.
19. Prove: If $f : X \to Y$ and $g : Y \to Z$ are surjective, so is $f \circ g : X \to Z$.
20. The identity map $f(x) = x$.

21. If $f$ is bijective, it has an inverse,

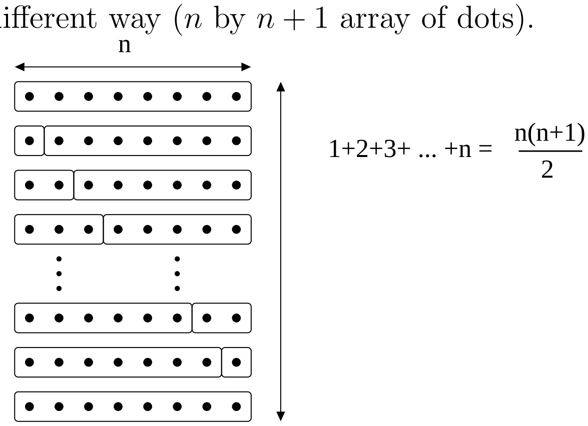
$$f^{-1} = \{(b, a) : (a, b) \in f\}.$$

Less formally, $f \circ f^{-} = f^{-1} \circ f$ is the identity.
22. Well ordering property of $\mathbb{N}$: Every non-empty subset of $\mathbb{N}$ has a least element.
23. Principle of Mathematical Induction: Let $S \subset \mathbb{N}$ so that
  (1) $1 \in S$
  (2) if $k \in S$ then $k + 1 \in S$
Then $S = \mathbb{N}$.
Second version; let $n_0 \in \mathbb{N}$ and let $P(n)$ be a statement for each natural number$n \geq n_0$. Suppse that:
  (1) The statement $P(n_0)$ is true.
  (2) For all $k \geq n_0$, the truth of $P(k)$ implies the truth of $P(k + 1)$.
Then $P(n)$ is true for all $n \geq n_0$.
24. Prove $1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$ by induction.
  Proof: Check for $n = 1$: $1 = \frac{1}{2} \cdot 1 \cdot 2$.
  Assume $1 + \ldots k = \frac{1}{2}k(k + 1)$. Then

$$1+\cdots+k+(k+1) = \frac{1}{2}k(k+1)+(k+1) = (\frac{1}{2}k+1)(k+1) = \frac{1}{2}(k+2)(k+1) = \frac{1}{2}(k+1)(k+2).$$

□

25. Prove same in different way ($n$ by $n + 1$ array of dots).




26. Prove every integer $\geq 2$ has a prime factorization.
: Induction hypothesis: Every integer $2 \leq k \leq n$ has a factorization in to primes.
  Check for $n = 2$. Easy.
  Assume for $n$. If $n + 1$ is prime, then we are done. Otherwise $n + 1 = a \cdot b$ and $a, b$ are both $< n + 1$. Hence ,$b$ are both $\leq n$, so they have prime factorizations

$$a = p_1 \cdots p_s, \quad b = q_1 \cdots q_t$$

, so $n + 1 == (p_1 \cdots p_s) \cdot (q_1 \cdots q_t)$ has a prime factorization.
  The factorization is unique (up to reordering), but this is harder to prove.

27. Prove that every integer $\geq 18$ is of the form $4a + 7b$ for non-negative integers $a$ and $b$,

Hypothesis: every integer $18 \leq k \leq n$ has this form.

Base case: $18 = 14 + 4 = 2 \cdot 7 + 4$.

Assume is true for all integers $k$ up to $n > 18$. Prove for $n + 1$. If $n + 1 \geq 22$ then $n - 4 \geq 18$, so $n - 4 = 7a + 4b$ for some $a, b$. Then

$$n = 4 + 7a + 4b = 7a + 4(b + 1).$$

If $18 < n + 1 < 22$ then $n + 1$ is 19, 20 or 21. But

$$19 = 7 + 12 = 7 + 3 \cdot 4,$$

$$20 = 4 \cdot 5$$

$$21 = 7 \cdot 3,$$

so the hypothesis is true in every case. $\square$