MAT 320, Prof. Bishop, Tuesday, Nov 23, 2021

Section 9.2: Tests for Absolute Convergence

8. **Raabe's Test:** Let (x_n) be a sequence of non-zero reals.

(a) if there are $a > 1$ and $K \in \mathbb{N}$ so that $n > K$ implies

$$\left|\frac{x_{n+1}}{x_n}\right| \leq 1 - \frac{a}{n}.$$

then $\sum x_n$ is absolutely convergent.

(b) if there are $a \leq$ and $K \in \mathbb{N}$ so that $n > K$ implies

$$\left|\frac{x_{n+1}}{x_n}\right| \geq 1 - \frac{a}{n}.$$

then $\sum x_n$ is absolutely convergent.

Proof:

(a) Rearranging, We get

$$(k-1)|x_k) - k|x_{k+1}| \geq (a-1)|x_k|.$$

for $k \geq K$. Hence $k|x_{k+1}|$ is decreasing. If we note the left sides telescopes,

$$(K-1)|K_k) - n|x_{n+1}| \geq (a-1)(|x_K| + \cdots + |x_n|).$$

This shows the partial sums are bounded, and hence the series $\sum x_n$ converges absolutely.

(b) The same reasoning shows $k|x_{k+1}|$ is eventually increasing and hence x_n is bounded below by C/n. Since the harmonic series diverges, so does (x_n).

10. **Corollary:** Assume

$$a = \lim(n(1 - \frac{|x_{n+1}|}{|x_n|}))$$

exists. If $a > 1$ then $\sum x_n$ is absolutely convergent, and if $a < 1$ is it is not absolutely convergent.

Proof: Suppose the limit exists and is > 1. If $1 < a_1 < a$ then $a_1 < n(1 - |x_{n+1}/x_n|)$ for sufficiently large n, so $|x_{n+1}/x_n| < 1 - a_1/n$ and Raabe's test applies.

If $a < 1$, the argument is similar.

Section 9.3: Test for Non-absolute Convergence

1. **Defn:** A non-zero series (x_n) is **alternating** if the x_{n+1} has the opposite sign to x_n.

2. **Alternating Series Test:** If (z_n) decreases to 0 then the alternating series $\sum(-1)^n z_n$ converges.

Proof: Note that

$$s_{2n} = (z_1 - z_2) + \ldots (z_{2n-1} - z_{2n}),$$

is increasing and bounded above by

$$s_{2n} = z_1 - (z_2 - z_3) + \cdots - z_{2n} \leq z_1$$

Hence the even partial sums converge to a limit L by the Monotone convergence theorem.

A similar argument show the odd partial sums are decreasing and bounded, so converge to some M. But

$$|s_{2n} - s_{2n+1}| = |z_{2n+1}| \to 0$$

so the two sequence have the same limit, hence the whole sequence converges.

3. **Abel's lemma:** Let (x_n), (y_n) be sequences in $\mathbb{R}$ and let $s_n = \sum_{k=1}^{n} y_k$, with $s_0 = 0$. If $m > n$, then

$$\sum_{k=n+1}^{m} x_k y_k = (x_n s_n - x_{n+1} s_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) s_k.$$

Proof: Since $y_k = s_k - s_{k-1}$, the left side is

$$\begin{aligned} \sum_{k=n+1}^{m} x_k (s_k - s_{k-1}) \\ &= (x_{n+1} s_{n+1} - x_{n+1} s_n) + (x_{n+2} s_{n+2} - x_{n+2} s_{n+1}) + \\ &\quad \ldots (x_m s_m - x_m s_{m-1}), \end{aligned}$$

which is the right hand side.

4. **Dirichlet's Test:** If (x_n) decreases to 0 and if the partial sums (s_n) of $\sum y_n$ are bounded, then $\sum x_n y_n$ is convergent.

Proof: Suppose $|s_n| \leq B$. Since $x_k - x_{k+1} \geq 0$,

$$\begin{aligned}
|\sum_{k=n+1}^{m} x_k y_k| &\leq (x_m + x_{n+1})B + \sum_{k=n+1}^{m-1} (x_k - x_{k+1})B \\
&= (x_m + x_{n+1})B + B \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) \\
&\leq B[(x_m + x_{n+1}) + (x_{n+1} - x_m)] \\
&= 2x_{n+1}B.
\end{aligned}$$

Since $x_n \to 0$, the series satisfies the Cauchy criterion and converges.

Special case is alternating series $y_n = (-1)^n$.

5. Example: recall

$$\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha+\beta) - \sin(\beta-\alpha)).$$

Hence

$$2\sin(\frac{x}{2})(\cos kx) = \sin(k+\frac{1}{2})x - \sin(k-\frac{1}{2})x,$$

$$2\sin(\frac{x}{2})(\cos x + \cdots + \cos nx) = \sin(n+\frac{1}{2})x - \sin\frac{x}{2},$$

implies

$$|\cos x + \cdots + \cos nx| \le \frac{1}{\sin(x/2)},$$

so if a_n decreases to 0,

$$\sum a_n \cos(nx)$$

converges for $x \neq 2\pi n$. Similar result for $\sum a_n \sin nx$.

Section 9.4: Series of functions.

1. **Defn:** If (f_n) is a sequence of functions, let (s_n) denote the sequence of partial sums

$$s_n(x) = f_1(x) + \cdots + f_n(x).$$

If the partial sums converge (pointwise) to f we say $\sum f_n$ converges to f.

2. Define absolutely convergent or uniformly convergent in obvious way.

3. **Theorem 9.4.2:** If (f_n) are all continuous and converge uniformly to f, then f is continuous.

3. **Theorem 9.4.3:** If (f_n) are all Riemann integrable on $J = [a, b]$ and $\sum f_n$ converges uniformly to f, then

$$\int f = \sum \int f_n.$$

3. **Theorem 9.4.4:** If (f_n) are all differentiable on $J = [a, b]$, that $\sum f_n$ converges at some point of J and that $\sum f_n'$ converges uniformly on J. Then there is a f so that $f - \sum f_n$ and $f' = \sum f_n'$.

4. **Theorem 9.4.5:** If (f_n) are functions $D \to \mathbb{R}$. The series $\sum f_n$ is uniformly convergent iff for all $\epsilon > 0$ there is a M so that $n > M$ implies

$$|f_{n+1}(x) + \cdots + f_m(x)| < \epsilon,$$

for all $x \in D$.

This is just the Cauchy criteria for uniform convergence.

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5. **Weierstrass M-test:** Let M_n be real numbers so that $|f_n(x)| \leq M_n$ for all $x \in D$. If $\sum M_n$ converges, then $\sum f_n$ is uniformly convergent.

6. **Defn:** a power series around $c \in \mathbb{R}$ is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n.$$

Most common case is $c = 0$.

7. A power series may converge only at c, on an interval $(c - r, c + r)$ or on all on $\mathbb{R}$.The limit of a convergent power series is called analytic.

8. Define

$$\rho = \limsup |a_n|^{1/n}.$$

The radius of convergence is $R = 0$ if $\rho = \infty$, $1/\rho$ if $0 < \rho < \infty$ and $R = \infty$ if $\rho = 0$.

9. **Cauchy-Hadamard:** If R is the radius of convergence of a power series $\sum a_n x^n$, then the series is absolutely convergent for $|x| < R$ and divergent for $|x| > R$.

10. All behaviors possible on boundary

$$\sum x^n, \qquad \sum \frac{1}{n} x^2, \qquad \sum \frac{1}{n^2} x^n.$$

11. **Theorem 9.4.10:** If $\sum a_n x^n$ has radius on convergence R then it converges uniformly on any closed, bounded interval $K \subset (-R, R)$.

12. **Theorem 9.4.11:** The limit of a power series is continuous on its interval of convergence. On any closed bounded subinterval, it can be integrated term-by-term .

13. **Theorem 9.4.12:** The limit of a power series is differentiable on its open interval of convergence. If $f(x) = \sum a_n x_n$ then

$$f'(x) = \sum n a_n x^{n-1}.$$

and both functions have the same radius of convergence.

14. **Theorem 9.4.13:** If $\sum a_n x_n$ and $\sum b_n x_n$ converge to the same function on the same interval $(-r, r)$, then $a_n = b_n$ for all n.

Proof: by above $a_n = f^{(n)}(0)/n! = b_n$.

15. If f has infinitely many derivatives, its Taylor series is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

If a power series converges to f it is the Taylor series of f.

But, just because a function is infinitely differentiable, its Taylor series need not converge to it. If the Taylor series of f does converge to f, then f is called analytic.

It is possible the Taylor series converges to the "wrong" function.

16. Examples:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$\exp x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

17. The **Fourier series** associated to f on $[-\pi, \pi]$ is

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)dx,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)dx.$$

Fourier series of continuous function need not converge to f everywhere.

Famous (and hard) theorem of Lennart Carleson says it does converge to f except on set of zero length.

If f is “nice”, e.g., Lipschitz or Hölder, then Fourier series does converge to f everywhere.

Study of Fourier series is vast topic. Many, many generalizations.