

7.2 | Calculus of Parametric Curves

Learning Objectives

- 7.2.1** Determine derivatives and equations of tangents for parametric curves.
- 7.2.2** Find the area under a parametric curve.
- 7.2.3** Use the equation for arc length of a parametric curve.
- 7.2.4** Apply the formula for surface area to a volume generated by a parametric curve.

Now that we have introduced the concept of a parameterized curve, our next step is to learn how to work with this concept in the context of calculus. For example, if we know a parameterization of a given curve, is it possible to calculate the slope of a tangent line to the curve? How about the arc length of the curve? Or the area under the curve?

Another scenario: Suppose we would like to represent the location of a baseball after the ball leaves a pitcher's hand. If the position of the baseball is represented by the plane curve $(x(t), y(t))$, then we should be able to use calculus to find the speed of the ball at any given time. Furthermore, we should be able to calculate just how far that ball has traveled as a function of time.

Derivatives of Parametric Equations

We start by asking how to calculate the slope of a line tangent to a parametric curve at a point. Consider the plane curve defined by the parametric equations

$$x(t) = 2t + 3, \quad y(t) = 3t - 4, \quad -2 \leq t \leq 3.$$

The graph of this curve appears in **Figure 7.16**. It is a line segment starting at $(-1, -10)$ and ending at $(9, 5)$.

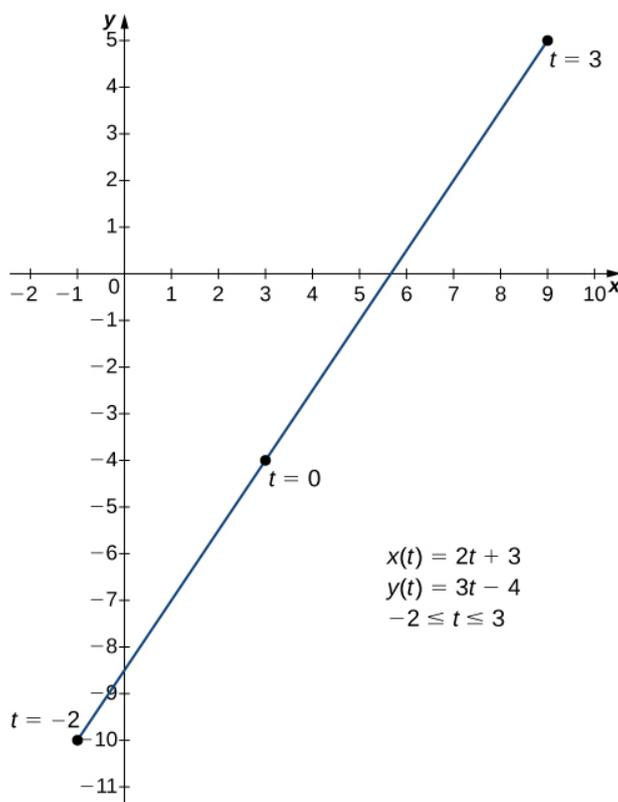


Figure 7.16 Graph of the line segment described by the given parametric equations.

We can eliminate the parameter by first solving the equation $x(t) = 2t + 3$ for t :

$$\begin{aligned}x(t) &= 2t + 3 \\x - 3 &= 2t \\t &= \frac{x - 3}{2}.\end{aligned}$$

Substituting this into $y(t)$, we obtain

$$\begin{aligned}y(t) &= 3t - 4 \\y &= 3\left(\frac{x - 3}{2}\right) - 4 \\y &= \frac{3x}{2} - \frac{9}{2} - 4 \\y &= \frac{3x}{2} - \frac{17}{2}.\end{aligned}$$

The slope of this line is given by $\frac{dy}{dx} = \frac{3}{2}$. Next we calculate $x'(t)$ and $y'(t)$. This gives $x'(t) = 2$ and $y'(t) = 3$. Notice that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3}{2}$. This is no coincidence, as outlined in the following theorem.

Theorem 7.1: Derivative of Parametric Equations

Consider the plane curve defined by the parametric equations $x = x(t)$ and $y = y(t)$. Suppose that $x'(t)$ and $y'(t)$ exist, and assume that $x'(t) \neq 0$. Then the derivative $\frac{dy}{dx}$ is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}. \quad (7.1)$$

Proof

This theorem can be proven using the Chain Rule. In particular, assume that the parameter t can be eliminated, yielding a differentiable function $y = F(x)$. Then $y(t) = F(x(t))$. Differentiating both sides of this equation using the Chain Rule yields

$$y'(t) = F'(x(t))x'(t),$$

so

$$F'(x(t)) = \frac{y'(t)}{x'(t)}.$$

But $F'(x(t)) = \frac{dy}{dx}$, which proves the theorem.

□

Equation 7.1 can be used to calculate derivatives of plane curves, as well as critical points. Recall that a critical point of a differentiable function $y = f(x)$ is any point $x = x_0$ such that either $f'(x_0) = 0$ or $f'(x_0)$ does not exist. **Equation 7.1** gives a formula for the slope of a tangent line to a curve defined parametrically regardless of whether the curve can be described by a function $y = f(x)$ or not.

Example 7.4

Finding the Derivative of a Parametric Curve

Calculate the derivative $\frac{dy}{dx}$ for each of the following parametrically defined plane curves, and locate any critical points on their respective graphs.

- $x(t) = t^2 - 3$, $y(t) = 2t - 1$, $-3 \leq t \leq 4$
- $x(t) = 2t + 1$, $y(t) = t^3 - 3t + 4$, $-2 \leq t \leq 5$
- $x(t) = 5 \cos t$, $y(t) = 5 \sin t$, $0 \leq t \leq 2\pi$

Solution

- To apply **Equation 7.1**, first calculate $x'(t)$ and $y'(t)$:

$$\begin{aligned}x'(t) &= 2t \\y'(t) &= 2.\end{aligned}$$

Next substitute these into the equation:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ \frac{dy}{dx} &= \frac{2}{2t} \\ \frac{dy}{dx} &= \frac{1}{t}.\end{aligned}$$

This derivative is undefined when $t = 0$. Calculating $x(0)$ and $y(0)$ gives $x(0) = (0)^2 - 3 = -3$ and $y(0) = 2(0) - 1 = -1$, which corresponds to the point $(-3, -1)$ on the graph. The graph of this curve is a parabola opening to the right, and the point $(-3, -1)$ is its vertex as shown.

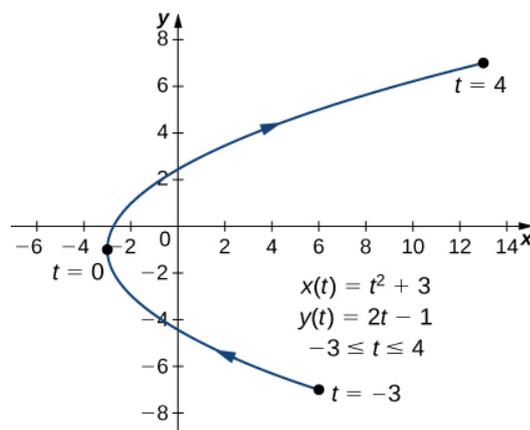


Figure 7.17 Graph of the parabola described by parametric equations in part a.

- To apply **Equation 7.1**, first calculate $x'(t)$ and $y'(t)$:

$$\begin{aligned}x'(t) &= 2 \\y'(t) &= 3t^2 - 3.\end{aligned}$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2}$$

This derivative is zero when $t = \pm 1$. When $t = -1$ we have

$$x(-1) = 2(-1) + 1 = -1 \text{ and } y(-1) = (-1)^3 - 3(-1) + 4 = -1 + 3 + 4 = 6,$$

which corresponds to the point $(-1, 6)$ on the graph. When $t = 1$ we have

$$x(1) = 2(1) + 1 = 3 \text{ and } y(1) = (1)^3 - 3(1) + 4 = 1 - 3 + 4 = 2,$$

which corresponds to the point $(3, 2)$ on the graph. The point $(3, 2)$ is a relative minimum and the point $(-1, 6)$ is a relative maximum, as seen in the following graph.

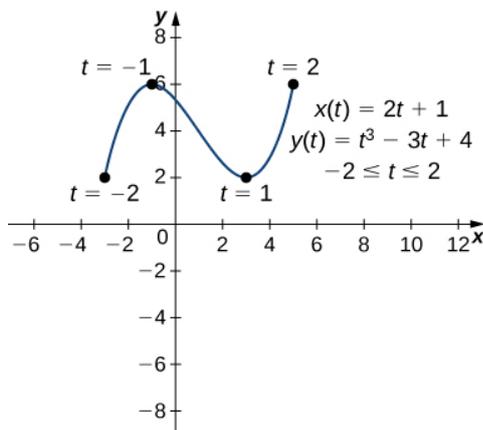


Figure 7.18 Graph of the curve described by parametric equations in part b.

- c. To apply **Equation 7.1**, first calculate $x'(t)$ and $y'(t)$:

$$x'(t) = -5 \sin t$$

$$y'(t) = 5 \cos t.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{5 \cos t}{-5 \sin t}$$

$$\frac{dy}{dx} = -\cot t.$$

This derivative is zero when $\cos t = 0$ and is undefined when $\sin t = 0$. This gives $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2},$ and 2π as critical points for t . Substituting each of these into $x(t)$ and $y(t)$, we obtain

t	$x(t)$	$y(t)$
0	5	0
$\frac{\pi}{2}$	0	5
π	-5	0
$\frac{3\pi}{2}$	0	-5
2π	5	0

These points correspond to the sides, top, and bottom of the circle that is represented by the parametric equations (**Figure 7.19**). On the left and right edges of the circle, the derivative is undefined, and on the top and bottom, the derivative equals zero.

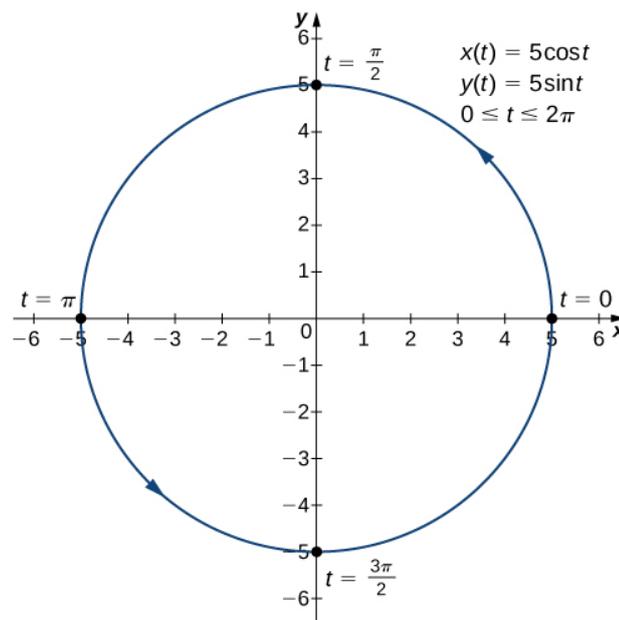


Figure 7.19 Graph of the curve described by parametric equations in part c.



7.4 Calculate the derivative dy/dx for the plane curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad -2 \leq t \leq 3$$

and locate any critical points on its graph.

Example 7.5

Finding a Tangent Line

Find the equation of the tangent line to the curve defined by the equations

$$x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad -3 \leq t \leq 4 \text{ when } t = 2.$$

Solution

First find the slope of the tangent line using **Equation 7.1**, which means calculating $x'(t)$ and $y'(t)$:

$$x'(t) = 2t$$

$$y'(t) = 2.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{2}{2t}$$

$$\frac{dy}{dx} = \frac{1}{t}.$$

When $t = 2$, $\frac{dy}{dx} = \frac{1}{2}$, so this is the slope of the tangent line. Calculating $x(2)$ and $y(2)$ gives

$$x(2) = (2)^2 - 3 = 1 \text{ and } y(2) = 2(2) - 1 = 3,$$

which corresponds to the point $(1, 3)$ on the graph (**Figure 7.20**). Now use the point-slope form of the equation of a line to find the equation of the tangent line:

$$y - y_0 = m(x - x_0)$$

$$y - 3 = \frac{1}{2}(x - 1)$$

$$y - 3 = \frac{1}{2}x - \frac{1}{2}$$

$$y = \frac{1}{2}x + \frac{5}{2}.$$

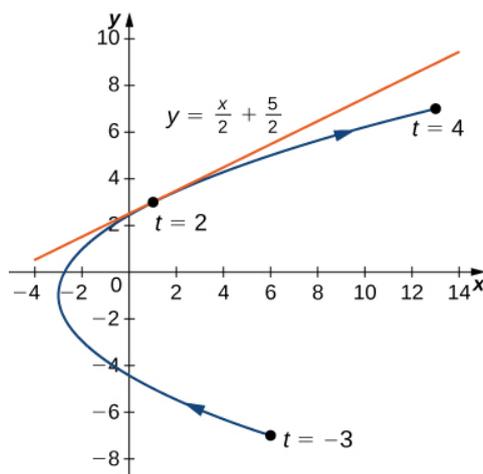


Figure 7.20 Tangent line to the parabola described by the given parametric equations when $t = 2$.



7.5 Find the equation of the tangent line to the curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad -2 \leq t \leq 3 \text{ when } t = 5.$$

Second-Order Derivatives

Our next goal is to see how to take the second derivative of a function defined parametrically. The second derivative of a function $y = f(x)$ is defined to be the derivative of the first derivative; that is,

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left[\frac{dy}{dx} \right].$$

Since $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, we can replace the y on both sides of this equation with $\frac{dy}{dx}$. This gives us

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{(d/dt)(dy/dx)}{dx/dt}. \quad (7.2)$$

If we know dy/dx as a function of t , then this formula is straightforward to apply.

Example 7.6

Finding a Second Derivative

Calculate the second derivative $d^2 y/dx^2$ for the plane curve defined by the parametric equations $x(t) = t^2 - 3$, $y(t) = 2t - 1$, $-3 \leq t \leq 4$.

Solution

From **Example 7.4** we know that $\frac{dy}{dx} = \frac{2}{2t} = \frac{1}{t}$. Using **Equation 7.2**, we obtain

$$\frac{d^2 y}{dx^2} = \frac{(d/dt)(dy/dx)}{dx/dt} = \frac{(d/dt)(1/t)}{2t} = \frac{-t^{-2}}{2t} = -\frac{1}{2t^3}.$$



7.6 Calculate the second derivative $d^2 y/dx^2$ for the plane curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad -2 \leq t \leq 3$$

and locate any critical points on its graph.

Integrals Involving Parametric Equations

Now that we have seen how to calculate the derivative of a plane curve, the next question is this: How do we find the area under a curve defined parametrically? Recall the cycloid defined by the equations $x(t) = t - \sin t$, $y(t) = 1 - \cos t$.

Suppose we want to find the area of the shaded region in the following graph.

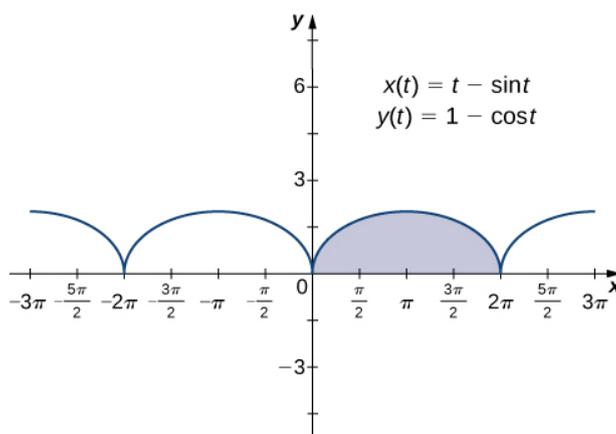


Figure 7.21 Graph of a cycloid with the arch over $[0, 2\pi]$ highlighted.

To derive a formula for the area under the curve defined by the functions

$$x = x(t), \quad y = y(t), \quad a \leq t \leq b,$$

we assume that $x(t)$ is differentiable and start with an equal partition of the interval $a \leq t \leq b$. Suppose $t_0 = a < t_1 < t_2 < \dots < t_n = b$ and consider the following graph.

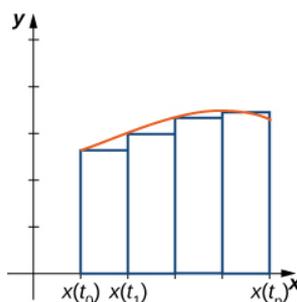


Figure 7.22 Approximating the area under a parametrically defined curve.

We use rectangles to approximate the area under the curve. The height of a typical rectangle in this parametrization is $y(x(\bar{t}_i))$ for some value \bar{t}_i in the i th subinterval, and the width can be calculated as $x(t_i) - x(t_{i-1})$. Thus the area of the i th rectangle is given by

$$A_i = y(x(\bar{t}_i))(x(t_i) - x(t_{i-1})).$$

Then a Riemann sum for the area is

$$A_n = \sum_{i=1}^n y(x(\bar{t}_i))(x(t_i) - x(t_{i-1})).$$

Multiplying and dividing each area by $t_i - t_{i-1}$ gives

$$A_n = \sum_{i=1}^n y(x(\bar{t}_i)) \left(\frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}} \right) (t_i - t_{i-1}) = \sum_{i=1}^n y(x(\bar{t}_i)) \left(\frac{x(t_i) - x(t_{i-1})}{\Delta t} \right) \Delta t.$$

Taking the limit as n approaches infinity gives

$$A = \lim_{n \rightarrow \infty} A_n = \int_a^b y(t)x'(t) dt.$$

This leads to the following theorem.

Theorem 7.2: Area under a Parametric Curve

Consider the non-self-intersecting plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad a \leq t \leq b$$

and assume that $x(t)$ is differentiable. The area under this curve is given by

$$A = \int_a^b y(t)x'(t) dt. \quad (7.3)$$

Example 7.7

Finding the Area under a Parametric Curve

Find the area under the curve of the cycloid defined by the equations

$$x(t) = t - \sin t, \quad y(t) = 1 - \cos t, \quad 0 \leq t \leq 2\pi.$$

Solution

Using **Equation 7.3**, we have

$$\begin{aligned} A &= \int_a^b y(t)x'(t) dt \\ &= \int_0^{2\pi} (1 - \cos t)(1 - \cos t) dt \\ &= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t) dt \\ &= \int_0^{2\pi} \left(1 - 2\cos t + \frac{1 + \cos 2t}{2}\right) dt \\ &= \int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \frac{\cos 2t}{2}\right) dt \\ &= \left.\frac{3t}{2} - 2\sin t + \frac{\sin 2t}{4}\right|_0^{2\pi} \\ &= 3\pi. \end{aligned}$$



7.7 Find the area under the curve of the hypocycloid defined by the equations

$$x(t) = 3\cos t + \cos 3t, \quad y(t) = 3\sin t - \sin 3t, \quad 0 \leq t \leq \pi.$$

Arc Length of a Parametric Curve

In addition to finding the area under a parametric curve, we sometimes need to find the arc length of a parametric curve. In the case of a line segment, arc length is the same as the distance between the endpoints. If a particle travels from point A to point B along a curve, then the distance that particle travels is the arc length. To develop a formula for arc length, we start with an approximation by line segments as shown in the following graph.

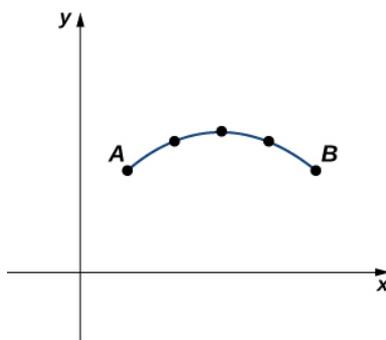


Figure 7.23 Approximation of a curve by line segments.

Given a plane curve defined by the functions $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, we start by partitioning the interval $[a, b]$ into n equal subintervals: $t_0 = a < t_1 < t_2 < \dots < t_n = b$. The width of each subinterval is given by $\Delta t = (b - a)/n$. We can calculate the length of each line segment:

$$d_1 = \sqrt{(x(t_1) - x(t_0))^2 + (y(t_1) - y(t_0))^2}$$

$$d_2 = \sqrt{(x(t_2) - x(t_1))^2 + (y(t_2) - y(t_1))^2} \text{ etc.}$$

Then add these up. We let s denote the exact arc length and s_n denote the approximation by n line segments:

$$s \approx \sum_{k=1}^n s_k = \sum_{k=1}^n \sqrt{(x(t_k) - x(t_{k-1}))^2 + (y(t_k) - y(t_{k-1}))^2}. \quad (7.4)$$

If we assume that $x(t)$ and $y(t)$ are differentiable functions of t , then the Mean Value Theorem (**Introduction to the Applications of Derivatives** (<http://cnx.org/content/m53602/latest/>)) applies, so in each subinterval $[t_{k-1}, t_k]$ there exist \hat{t}_k and \tilde{t}_k such that

$$x(t_k) - x(t_{k-1}) = x'(\hat{t}_k)(t_k - t_{k-1}) = x'(\hat{t}_k)\Delta t$$

$$y(t_k) - y(t_{k-1}) = y'(\tilde{t}_k)(t_k - t_{k-1}) = y'(\tilde{t}_k)\Delta t.$$

Therefore **Equation 7.4** becomes

$$s \approx \sum_{k=1}^n s_k$$

$$= \sum_{k=1}^n \sqrt{(x'(\hat{t}_k)\Delta t)^2 + (y'(\tilde{t}_k)\Delta t)^2}$$

$$= \sum_{k=1}^n \sqrt{(x'(\hat{t}_k))^2 (\Delta t)^2 + (y'(\tilde{t}_k))^2 (\Delta t)^2}$$

$$= \left(\sum_{k=1}^n \sqrt{(x'(\hat{t}_k))^2 + (y'(\tilde{t}_k))^2} \right) \Delta t.$$

This is a Riemann sum that approximates the arc length over a partition of the interval $[a, b]$. If we further assume that the derivatives are continuous and let the number of points in the partition increase without bound, the approximation approaches the exact arc length. This gives

$$\begin{aligned}
 s &= \lim_{n \rightarrow \infty} \sum_{k=1}^n s_k \\
 &= \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \sqrt{(x'(\hat{t}_k))^2 + (y'(\tilde{t}_k))^2} \right) \Delta t \\
 &= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt.
 \end{aligned}$$

When taking the limit, the values of \hat{t}_k and \tilde{t}_k are both contained within the same ever-shrinking interval of width Δt , so they must converge to the same value.

We can summarize this method in the following theorem.

Theorem 7.3: Arc Length of a Parametric Curve

Consider the plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad t_1 \leq t \leq t_2$$

and assume that $x(t)$ and $y(t)$ are differentiable functions of t . Then the arc length of this curve is given by

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt. \quad (7.5)$$

At this point a side derivation leads to a previous formula for arc length. In particular, suppose the parameter can be eliminated, leading to a function $y = F(x)$. Then $y(t) = F(x(t))$ and the Chain Rule gives $y'(t) = F'(x(t))x'(t)$. Substituting this into **Equation 7.5** gives

$$\begin{aligned}
 s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
 &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(F'(x)\frac{dx}{dt}\right)^2} dt \\
 &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 (1 + (F'(x))^2)} dt \\
 &= \int_{t_1}^{t_2} x'(t) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dt.
 \end{aligned}$$

Here we have assumed that $x'(t) > 0$, which is a reasonable assumption. The Chain Rule gives $dx = x'(t) dt$, and letting $a = x(t_1)$ and $b = x(t_2)$ we obtain the formula

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

which is the formula for arc length obtained in the **Introduction to the Applications of Integration**.

Example 7.8

Finding the Arc Length of a Parametric Curve

Find the arc length of the semicircle defined by the equations

$$x(t) = 3 \cos t, \quad y(t) = 3 \sin t, \quad 0 \leq t \leq \pi.$$

Solution

The values $t = 0$ to $t = \pi$ trace out the red curve in **Figure 7.23**. To determine its length, use **Equation 7.5**:

$$\begin{aligned} s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^\pi \sqrt{(-3 \sin t)^2 + (3 \cos t)^2} dt \\ &= \int_0^\pi \sqrt{9 \sin^2 t + 9 \cos^2 t} dt \\ &= \int_0^\pi \sqrt{9(\sin^2 t + \cos^2 t)} dt \\ &= \int_0^\pi 3 dt = 3t \Big|_0^\pi = 3\pi. \end{aligned}$$

Note that the formula for the arc length of a semicircle is πr and the radius of this circle is 3. This is a great example of using calculus to derive a known formula of a geometric quantity.

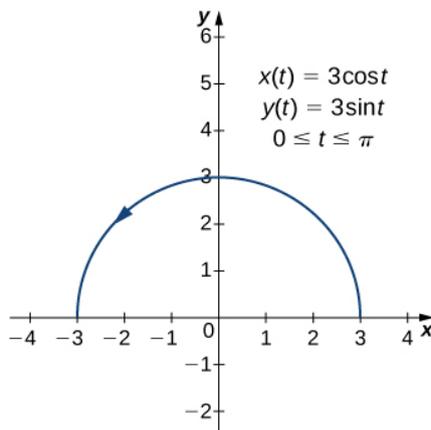


Figure 7.24 The arc length of the semicircle is equal to its radius times π .



7.8 Find the arc length of the curve defined by the equations

$$x(t) = 3t^2, \quad y(t) = 2t^3, \quad 1 \leq t \leq 3.$$

We now return to the problem posed at the beginning of the section about a baseball leaving a pitcher's hand. Ignoring the effect of air resistance (unless it is a curve ball!), the ball travels a parabolic path. Assuming the pitcher's hand is at the origin and the ball travels left to right in the direction of the positive x -axis, the parametric equations for this curve can be written as

$$x(t) = 140t, \quad y(t) = -16t^2 + 2t$$

where t represents time. We first calculate the distance the ball travels as a function of time. This distance is represented by the arc length. We can modify the arc length formula slightly. First rewrite the functions $x(t)$ and $y(t)$ using v as an independent variable, so as to eliminate any confusion with the parameter t :

$$x(v) = 140v, \quad y(v) = -16v^2 + 2v.$$

Then we write the arc length formula as follows:

$$\begin{aligned} s(t) &= \int_0^t \sqrt{\left(\frac{dx}{dv}\right)^2 + \left(\frac{dy}{dv}\right)^2} dv \\ &= \int_0^t \sqrt{140^2 + (-32v + 2)^2} dv. \end{aligned}$$

The variable v acts as a dummy variable that disappears after integration, leaving the arc length as a function of time t . To integrate this expression we can use a formula from **Appendix A**,

$$\int \sqrt{a^2 + u^2} du = \frac{u}{2} \sqrt{a^2 + u^2} + \frac{a^2}{2} \ln|u + \sqrt{a^2 + u^2}| + C.$$

We set $a = 140$ and $u = -32v + 2$. This gives $du = -32dv$, so $dv = -\frac{1}{32}du$. Therefore

$$\begin{aligned} \int \sqrt{140^2 + (-32v + 2)^2} dv &= -\frac{1}{32} \int \sqrt{a^2 + u^2} du \\ &= -\frac{1}{32} \left[\frac{(-32v + 2)}{2} \sqrt{140^2 + (-32v + 2)^2} \right. \\ &\quad \left. + \frac{140^2}{2} \ln|(-32v + 2) + \sqrt{140^2 + (-32v + 2)^2}| \right] + C \end{aligned}$$

and

$$\begin{aligned} s(t) &= -\frac{1}{32} \left[\frac{(-32t + 2)}{2} \sqrt{140^2 + (-32t + 2)^2} + \frac{140^2}{2} \ln|(-32t + 2) + \sqrt{140^2 + (-32t + 2)^2}| \right] \\ &\quad + \frac{1}{32} \left[\sqrt{140^2 + 2^2} + \frac{140^2}{2} \ln|2 + \sqrt{140^2 + 2^2}| \right] \\ &= \left(\frac{t}{2} - \frac{1}{32}\right) \sqrt{1024t^2 - 128t + 19604} - \frac{1225}{4} \ln|(-32t + 2) + \sqrt{1024t^2 - 128t + 19604}| \\ &\quad + \frac{\sqrt{19604}}{32} + \frac{1225}{4} \ln(2 + \sqrt{19604}). \end{aligned}$$

This function represents the distance traveled by the ball as a function of time. To calculate the speed, take the derivative of this function with respect to t . While this may seem like a daunting task, it is possible to obtain the answer directly from the Fundamental Theorem of Calculus:

$$\frac{d}{dx} \int_a^x f(u) du = f(x).$$

Therefore

$$\begin{aligned} s'(t) &= \frac{d}{dt}[s(t)] \\ &= \frac{d}{dt} \left[\int_0^t \sqrt{140^2 + (-32v + 2)^2} dv \right] \\ &= \sqrt{140^2 + (-32t + 2)^2} \\ &= \sqrt{1024t^2 - 128t + 19604} \\ &= 2\sqrt{256t^2 - 32t + 4901}. \end{aligned}$$

One third of a second after the ball leaves the pitcher's hand, the distance it travels is equal to

$$\begin{aligned}
 s\left(\frac{1}{3}\right) &= \left(\frac{1/3}{2} - \frac{1}{32}\right) \sqrt{1024\left(\frac{1}{3}\right)^2 - 128\left(\frac{1}{3}\right) + 19604} \\
 &\quad - \frac{1225}{4} \ln \left| \left(-32\left(\frac{1}{3}\right) + 2\right) + \sqrt{1024\left(\frac{1}{3}\right)^2 - 128\left(\frac{1}{3}\right) + 19604} \right| \\
 &\quad + \frac{\sqrt{19604}}{32} + \frac{1225}{4} \ln(2 + \sqrt{19604}) \\
 &\approx 46.69 \text{ feet.}
 \end{aligned}$$

This value is just over three quarters of the way to home plate. The speed of the ball is

$$s'\left(\frac{1}{3}\right) = 2 \sqrt{256\left(\frac{1}{3}\right)^2 - 16\left(\frac{1}{3}\right) + 4901} \approx 140.34 \text{ ft/s.}$$

This speed translates to approximately 95 mph—a major-league fastball.

Surface Area Generated by a Parametric Curve

Recall the problem of finding the surface area of a volume of revolution. In **Curve Length and Surface Area**, we derived a formula for finding the surface area of a volume generated by a function $y = f(x)$ from $x = a$ to $x = b$, revolved around the x -axis:

$$S = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} dx.$$

We now consider a volume of revolution generated by revolving a parametrically defined curve $x = x(t)$, $y = y(t)$, $a \leq t \leq b$ around the x -axis as shown in the following figure.

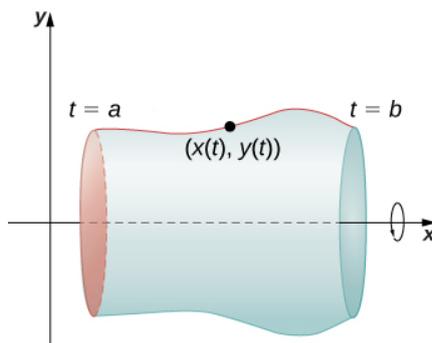


Figure 7.25 A surface of revolution generated by a parametrically defined curve.

The analogous formula for a parametrically defined curve is

$$S = 2\pi \int_a^b y(t) \sqrt{(x'(t))^2 + (y'(t))^2} dt \quad (7.6)$$

provided that $y(t)$ is not negative on $[a, b]$.

Example 7.9

Finding Surface Area

Find the surface area of a sphere of radius r centered at the origin.

Solution

We start with the curve defined by the equations

$$x(t) = r \cos t, \quad y(t) = r \sin t, \quad 0 \leq t \leq \pi.$$

This generates an upper semicircle of radius r centered at the origin as shown in the following graph.

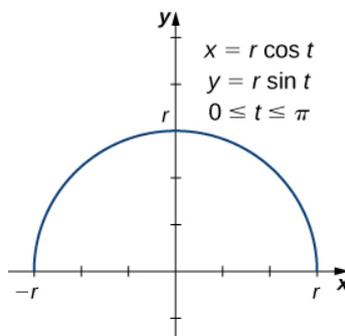


Figure 7.26 A semicircle generated by parametric equations.

When this curve is revolved around the x -axis, it generates a sphere of radius r . To calculate the surface area of the sphere, we use **Equation 7.6**:

$$\begin{aligned} S &= 2\pi \int_a^b y(t) \sqrt{(x'(t))^2 + (y'(t))^2} dt \\ &= 2\pi \int_0^\pi r \sin t \sqrt{(-r \sin t)^2 + (r \cos t)^2} dt \\ &= 2\pi \int_0^\pi r \sin t \sqrt{r^2 \sin^2 t + r^2 \cos^2 t} dt \\ &= 2\pi \int_0^\pi r \sin t \sqrt{r^2 (\sin^2 t + \cos^2 t)} dt \\ &= 2\pi \int_0^\pi r^2 \sin t dt \\ &= 2\pi r^2 (-\cos t) \Big|_0^\pi \\ &= 2\pi r^2 (-\cos \pi + \cos 0) \\ &= 4\pi r^2. \end{aligned}$$

This is, in fact, the formula for the surface area of a sphere.



7.9 Find the surface area generated when the plane curve defined by the equations

$$x(t) = t^3, \quad y(t) = t^2, \quad 0 \leq t \leq 1$$

is revolved around the x -axis.

7.2 EXERCISES

For the following exercises, each set of parametric equations represents a line. Without eliminating the parameter, find the slope of each line.

62. $x = 3 + t, y = 1 - t$

63. $x = 8 + 2t, y = 1$

64. $x = 4 - 3t, y = -2 + 6t$

65. $x = -5t + 7, y = 3t - 1$

For the following exercises, determine the slope of the tangent line, then find the equation of the tangent line at the given value of the parameter.

66. $x = 3 \sin t, y = 3 \cos t, t = \frac{\pi}{4}$

67. $x = \cos t, y = 8 \sin t, t = \frac{\pi}{2}$

68. $x = 2t, y = t^3, t = -1$

69. $x = t + \frac{1}{t}, y = t - \frac{1}{t}, t = 1$

70. $x = \sqrt{t}, y = 2t, t = 4$

For the following exercises, find all points on the curve that have the given slope.

71. $x = 4 \cos t, y = 4 \sin t, \text{ slope} = 0.5$

72. $x = 2 \cos t, y = 8 \sin t, \text{ slope} = -1$

73. $x = t + \frac{1}{t}, y = t - \frac{1}{t}, \text{ slope} = 1$

74. $x = 2 + \sqrt{t}, y = 2 - 4t, \text{ slope} = 0$

For the following exercises, write the equation of the tangent line in Cartesian coordinates for the given parameter t .

75. $x = e^{\sqrt{t}}, y = 1 - \ln t^2, t = 1$

76. $x = t \ln t, y = \sin^2 t, t = \frac{\pi}{4}$

77. $x = e^t, y = (t - 1)^2, \text{ at}(1, 1)$

78. For $x = \sin(2t), y = 2 \sin t$ where $0 \leq t < 2\pi$. Find all values of t at which a horizontal tangent line exists.

79. For $x = \sin(2t), y = 2 \sin t$ where $0 \leq t < 2\pi$. Find all values of t at which a vertical tangent line exists.

80. Find all points on the curve $x = 4 \cos(t), y = 4 \sin(t)$ that have the slope of $\frac{1}{2}$.

81. Find $\frac{dy}{dx}$ for $x = \sin(t), y = \cos(t)$.

82. Find the equation of the tangent line to $x = \sin(t), y = \cos(t)$ at $t = \frac{\pi}{4}$.

83. For the curve $x = 4t, y = 3t - 2$, find the slope and concavity of the curve at $t = 3$.

84. For the parametric curve whose equation is $x = 4 \cos \theta, y = 4 \sin \theta$, find the slope and concavity of the curve at $\theta = \frac{\pi}{4}$.

85. Find the slope and concavity for the curve whose equation is $x = 2 + \sec \theta, y = 1 + 2 \tan \theta$ at $\theta = \frac{\pi}{6}$.

86. Find all points on the curve $x = t + 4, y = t^3 - 3t$ at which there are vertical and horizontal tangents.

87. Find all points on the curve $x = \sec \theta, y = \tan \theta$ at which horizontal and vertical tangents exist.

For the following exercises, find d^2y/dx^2 .

88. $x = t^4 - 1, y = t - t^2$

89. $x = \sin(\pi t), y = \cos(\pi t)$

90. $x = e^{-t}, y = te^{2t}$

For the following exercises, find points on the curve at which tangent line is horizontal or vertical.

91. $x = t(t^2 - 3), y = 3(t^2 - 3)$

92. $x = \frac{3t}{1 + t^3}, y = \frac{3t^2}{1 + t^3}$

For the following exercises, find dy/dx at the value of the parameter.

93. $x = \cos t, y = \sin t, t = \frac{3\pi}{4}$

94. $x = \sqrt{t}$, $y = 2t + 4$, $t = 9$

95. $x = 4 \cos(2\pi s)$, $y = 3 \sin(2\pi s)$, $s = -\frac{1}{4}$

For the following exercises, find d^2y/dx^2 at the given point without eliminating the parameter.

96. $x = \frac{1}{2}t^2$, $y = \frac{1}{3}t^3$, $t = 2$

97. $x = \sqrt{t}$, $y = 2t + 4$, $t = 1$

98. Find t intervals on which the curve $x = 3t^2$, $y = t^3 - t$ is concave up as well as concave down.

99. Determine the concavity of the curve $x = 2t + \ln t$, $y = 2t - \ln t$.

100. Sketch and find the area under one arch of the cycloid $x = r(\theta - \sin \theta)$, $y = r(1 - \cos \theta)$.

101. Find the area bounded by the curve $x = \cos t$, $y = e^t$, $0 \leq t \leq \frac{\pi}{2}$ and the lines $y = 1$ and $x = 0$.

102. Find the area enclosed by the ellipse $x = a \cos \theta$, $y = b \sin \theta$, $0 \leq \theta < 2\pi$.

103. Find the area of the region bounded by $x = 2 \sin^2 \theta$, $y = 2 \sin^2 \theta \tan \theta$, for $0 \leq \theta \leq \frac{\pi}{2}$.

For the following exercises, find the area of the regions bounded by the parametric curves and the indicated values of the parameter.

104. $x = 2 \cot \theta$, $y = 2 \sin^2 \theta$, $0 \leq \theta \leq \pi$

105. **[T]**
 $x = 2a \cos t - a \cos(2t)$, $y = 2a \sin t - a \sin(2t)$, $0 \leq t < 2\pi$

106. **[T]** $x = a \sin(2t)$, $y = b \sin(t)$, $0 \leq t < 2\pi$ (the “hourglass”)

107. **[T]**
 $x = 2a \cos t - a \sin(2t)$, $y = b \sin t$, $0 \leq t < 2\pi$ (the “teardrop”)

For the following exercises, find the arc length of the curve on the indicated interval of the parameter.

108. $x = 4t + 3$, $y = 3t - 2$, $0 \leq t \leq 2$

109. $x = \frac{1}{3}t^3$, $y = \frac{1}{2}t^2$, $0 \leq t \leq 1$

110. $x = \cos(2t)$, $y = \sin(2t)$, $0 \leq t \leq \frac{\pi}{2}$

111. $x = 1 + t^2$, $y = (1 + t)^3$, $0 \leq t \leq 1$

112. $x = e^t \cos t$, $y = e^t \sin t$, $0 \leq t \leq \frac{\pi}{2}$ (express answer as a decimal rounded to three places)

113. $x = a \cos^3 \theta$, $y = a \sin^3 \theta$ on the interval $[0, 2\pi)$ (the hypocycloid)

114. Find the length of one arch of the cycloid $x = 4(t - \sin t)$, $y = 4(1 - \cos t)$.

115. Find the distance traveled by a particle with position (x, y) as t varies in the given time interval:
 $x = \sin^2 t$, $y = \cos^2 t$, $0 \leq t \leq 3\pi$.

116. Find the length of one arch of the cycloid $x = \theta - \sin \theta$, $y = 1 - \cos \theta$.

117. Show that the total length of the ellipse $x = 4 \sin \theta$, $y = 3 \cos \theta$ is

$$L = 16 \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta} d\theta, \quad \text{where } e = \frac{c}{a} \quad \text{and} \\ c = \sqrt{a^2 - b^2}.$$

118. Find the length of the curve $x = e^t - t$, $y = 4e^{t/2}$, $-8 \leq t \leq 3$.

For the following exercises, find the area of the surface obtained by rotating the given curve about the x -axis.

119. $x = t^3$, $y = t^2$, $0 \leq t \leq 1$

120. $x = a \cos^3 \theta$, $y = a \sin^3 \theta$, $0 \leq \theta \leq \frac{\pi}{2}$

121. **[T]** Use a CAS to find the area of the surface generated by rotating $x = t + t^3$, $y = t - \frac{1}{t^2}$, $1 \leq t \leq 2$ about the x -axis. (Answer to three decimal places.)

122. Find the surface area obtained by rotating $x = 3t^2$, $y = 2t^3$, $0 \leq t \leq 5$ about the y -axis.

123. Find the area of the surface generated by revolving $x = t^2$, $y = 2t$, $0 \leq t \leq 4$ about the x -axis.

124. Find the surface area generated by revolving $x = t^2$, $y = 2t^2$, $0 \leq t \leq 1$ about the y -axis.