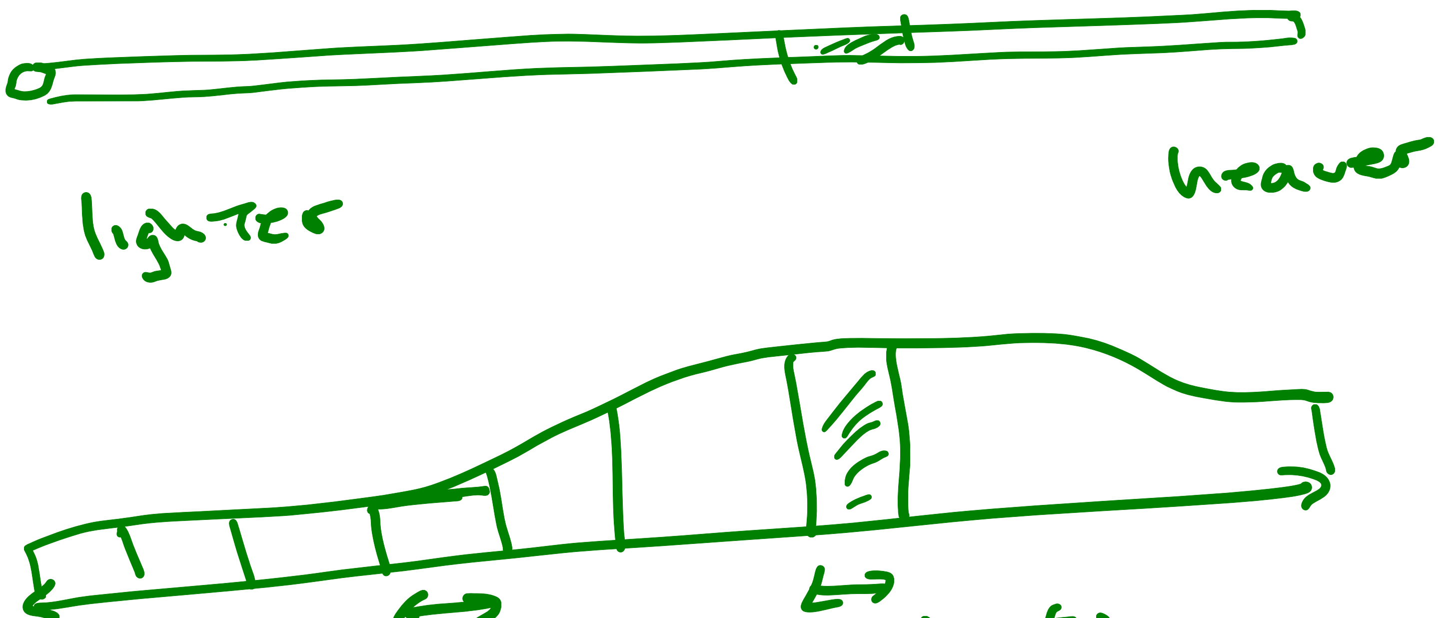
MAT 126.01, Prof. Bishop, Thursday, Oct 8, 2020
Section 2.5, Physical Applications (word problems)
Quiz 6 review

Mass of a rod:

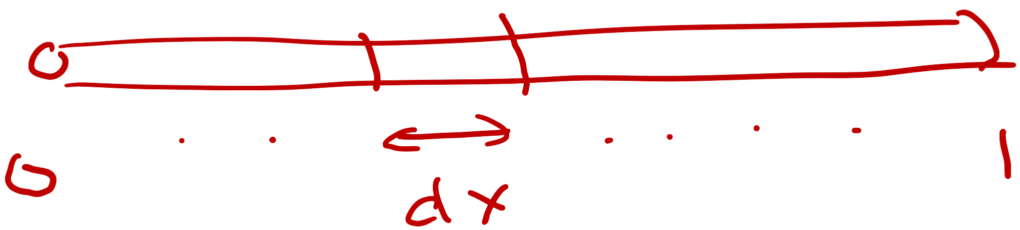
Given a thin rod on $[a, b]$ with density $\rho(x)$, the total mass is

$$\int_a^b \rho(x)dx.$$



$\rho(x) \cdot dx + \rho(x) \cdot dx + \ldots$

If a rod has mass density $\rho(x) = 1 + x^2$ grams per meter on a one meter long rod, what is the total mass?



$\rho(x) = 1 + x^2$

$$\int_0^1 (1+x^2)\,dx = x + \frac{1}{3}x^3 \Big|_0^1$$
$$= \left(1 + \frac{1}{3}1^3\right) - (0)$$
$$= 1 + \frac{1}{3}$$
$$= 4/3$$

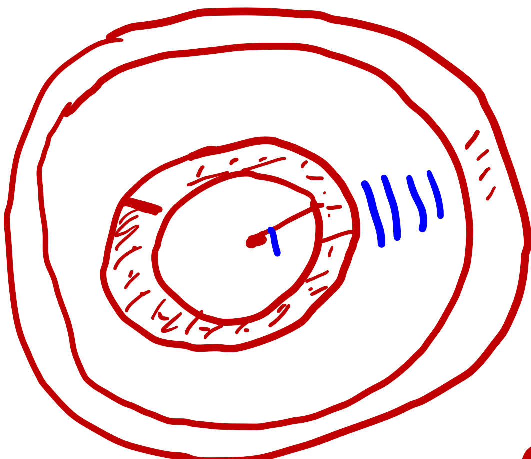
ρ = "rho"

P

Mass of a disk:

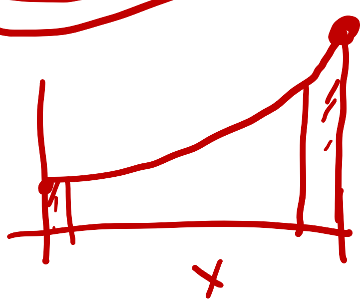
If a disk of radius r has density $\rho(x)$ at points distance x from its center, the mass of the disk is

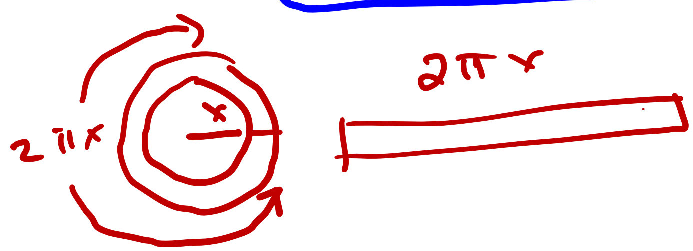
$$\int_0^r 2\pi x\rho(x)dx.$$



Area ring

$= 2\pi x \cdot dx$

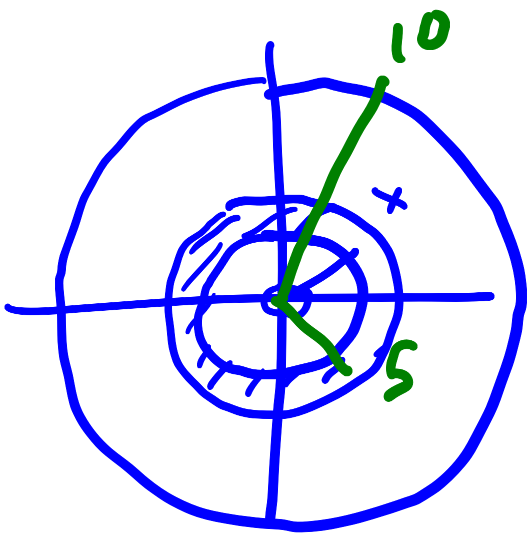




Statisticians estimate the population density is about $100,000/(1+x^2)$ people per square mile at x miles from the center of a certain city. How many people live within 10 miles of the city center?

What fraction lives within 5 miles of the center?

Midterm 2



$$\int_0^{10} P(x)\cdot 2\pi x\,dx$$

$$= 100,000 \cdot \int_0^{10} \frac{2\pi x\,dx}{1+x^2}$$

$$u = 1+x^2 \qquad du = 2x\,dx$$

$$= 100000\,\pi \int_0^{10} \frac{du}{u}$$

$$= 100,000\,\pi \ln u$$

$$\Big|_0^{10} \quad 5$$

$$= 100,000\,\pi \ln(1+x^2)$$

$$\boxed{= 100,000 \cdot \pi \ln(101)} \ldots$$

$$101 = 1+10^2$$

$$P(x) = \frac{100,000}{1+x^2}$$

5 miles:

$$= \boxed{100,000\,\pi \ln 26}$$

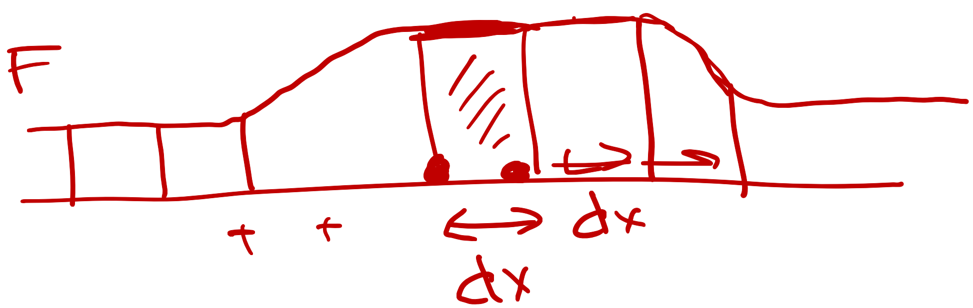
$$\text{ratio} = \ln 26 / \ln 101$$

Work done by a force:

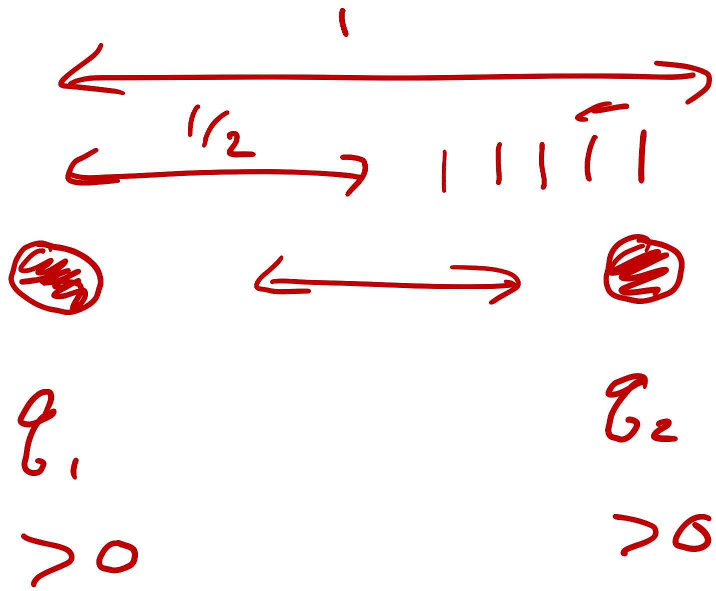
If a variable force $F(x)$ moves an object in the positive direction along the x-axis from a to b, the work done is

$$W = \int_a^b F(x)dx.$$

$W = F \cdot d$



Coulomb's Law says that two negatively charged particles repel each other with a force kq_1q_2/x^2 Newtons, where q_1, q_2 are the sizes of the charges, x is the distance between them, and k is Coulomb's constant.

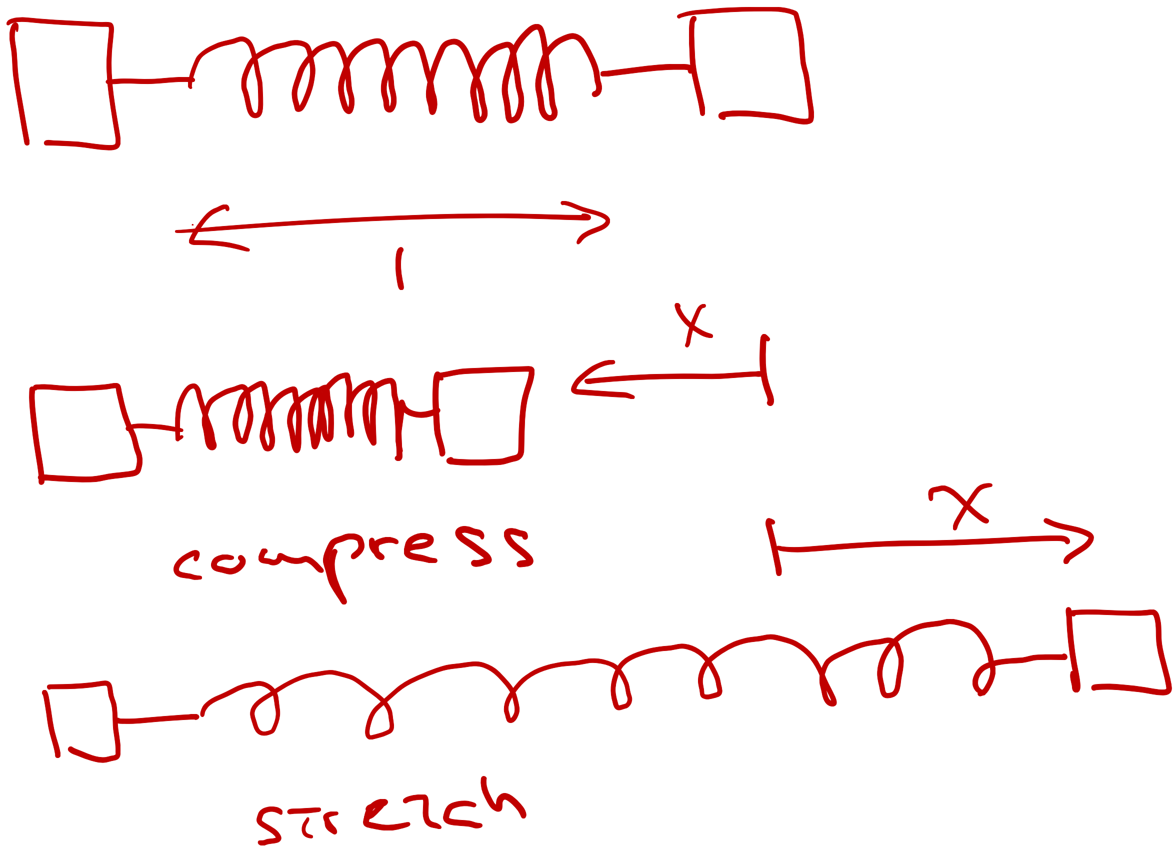


$$F \approx \frac{c}{x^2}$$

$$\int_{1/2}^{1} \frac{c}{x^2}\,dx \qquad c = k q_1 q_2$$

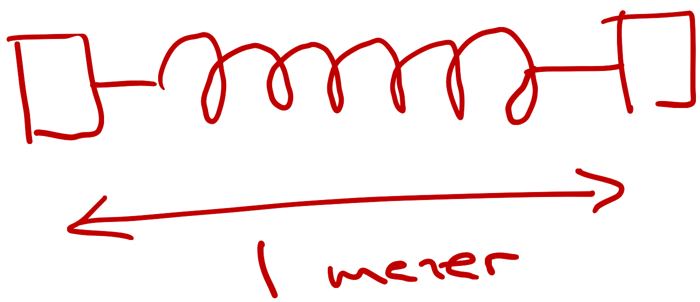
$$c \cdot -\frac{1}{x}\Big|_{1/2}^{1} = c\cdot\left(-1 - -\tfrac{1}{2}\right)$$
$$= c \cdot 1/2$$

Hooke's law: The force needed to compress or stretch a spring distance x from equilibrium is $F(x) = kx$ where k is constant depending on the spring.

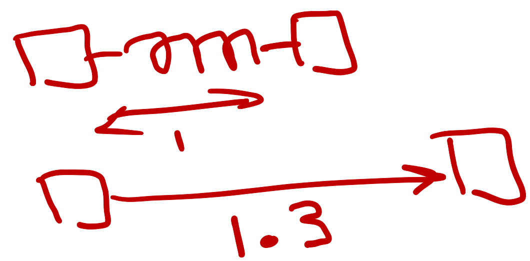


If 10 N of force compress a 1 meter spring by 20 centimeters, how much work is needed to stretch the spring by 30 centimeters?

Hint: first find the spring constant k.



20 cent = .2 meters



$F = k \cdot x$

$10 = k(.2)$?

$k = \frac{10}{.2} = 5 \cdot 10 = 50$

$$\int_0^{.3} kx\,dx$$

$$= \int_0^{.3} 50x\,dx$$

$$= 50 \left.\tfrac{1}{2}x^2\right|_0^{.3}$$

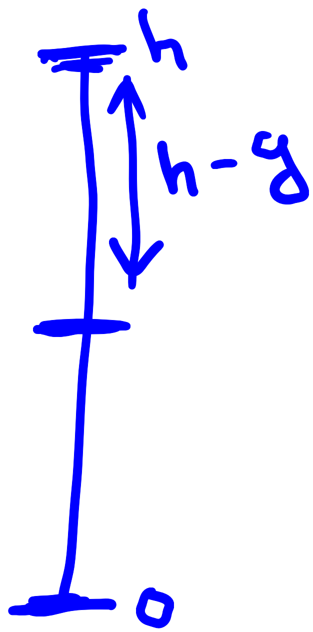
$$= 50 \cdot \tfrac{1}{2}(.3^2 - 0)$$

$$\boxed{= 25 \cdot (.09)}$$

Pumping out a tank:

If a tank contains a liquid that has to be pumped to a given height (maybe the top of the tank):

1. Sketch the tank
2. Cut into horizontal slices, compute the area
3. Multiply by the distance that slice is lifted.
4. Integrate between bottom and top levels.
5. Multiply by weight-density of the liquid



(vol slice × density)

× (distance to lift)

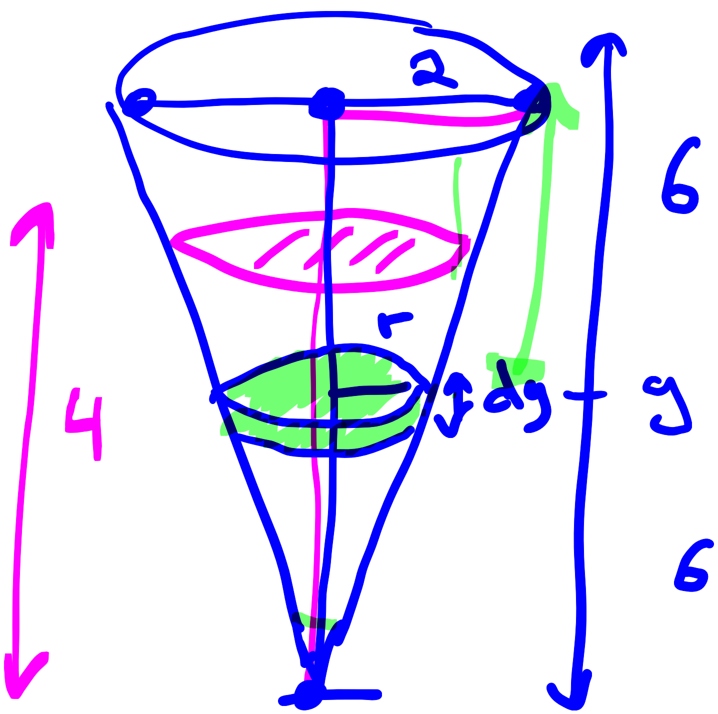
A water tank is shaped like a cone with the vertex down. The tank is 6 feet high and has diameter 4 feet at the top. If there is currently 4 feet of water in the tank, how much work is needed to pump all the water out of tank by lifting it over the top edge of the tank?

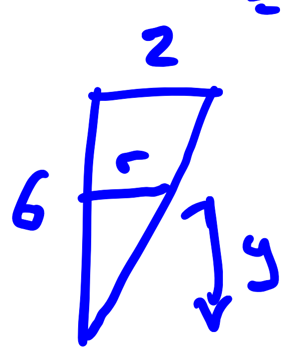
The weight density of water is 62.4 pounds per cubic foot or 9800 Newtons per meter cubed.

Quiz

vol = area disk · dy

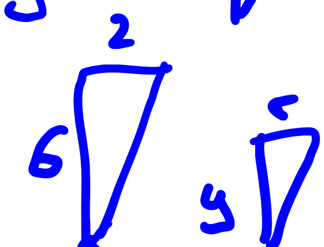
$= \pi r^2 \, dx$





Similar triangles

$$\boxed{\frac{2}{6} = \frac{r}{y}}$$



$$\frac{1}{3} y = r$$

$$W = \int_0^4 \pi \left(\frac{y}{3}\right)^2 (6-y) \, dy$$

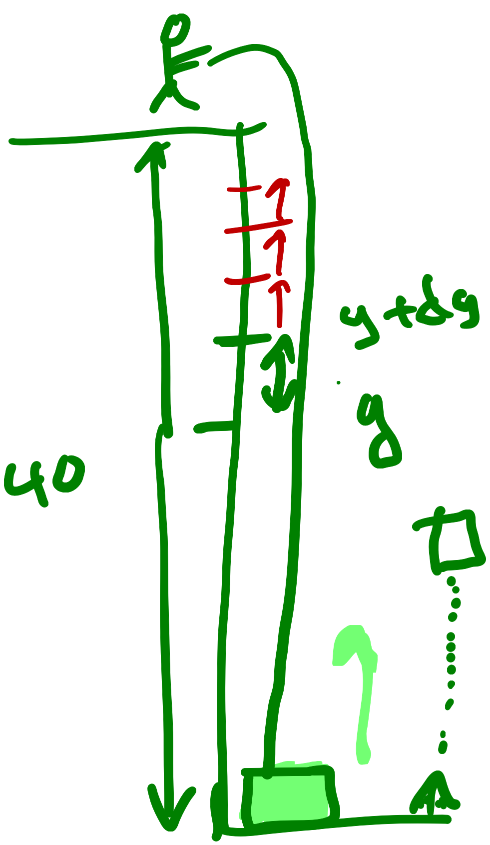
$$W = (62.4) \cdot \int_0^4 \ldots \, dy$$

Lifting a leaking bucket:

No leaking $\quad w = F \cdot d = 100 \cdot 40 = 4000$

A 100 lb bucket of sand is being lifted to the top of a 40 foot building at the rate of 10 feet per minute. 5 lbs of sand leaks out each minute. How much work (in foot-pounds) is done lifting the bucket to the roof?

Just give the integral. Ignore other factors, like the weight of the rope.



How heavy is bucket at height y?

Takes $y/10$ minutes to get to height y. At time $y/10$ the buckets weighs $100 - \underbrace{5 \cdot \frac{y}{10}}_{\text{lost sand}}$.

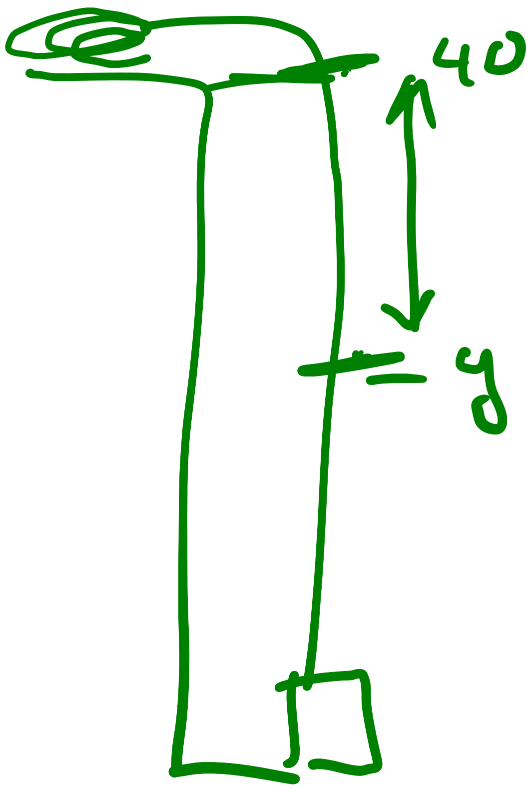
$$\boxed{\int_0^{40} \underbrace{\left(100 - \frac{y}{2}\right)}_{\text{weight}} \cdot \underbrace{dy}_{\text{dist}}}$$

$$= \left(100y - \frac{y^2}{4}\right)\Big|_0 = 4000 - \frac{1600}{4}$$

$$= 4000 - 400 = 3600$$

1 2

Suppose in the previous problem the rope weighs ∧ lb per foot. Now what integral gives the work done lifting the bucket?



$$\text{weight} = \underbrace{100 - \frac{y}{2}}_{\text{bucket}} + \underbrace{2(40-y)}_{\text{rope}}$$

$$\int_0^{40} \left(100 - \frac{y}{2} + 40 - y\right)$$

$\vdots$

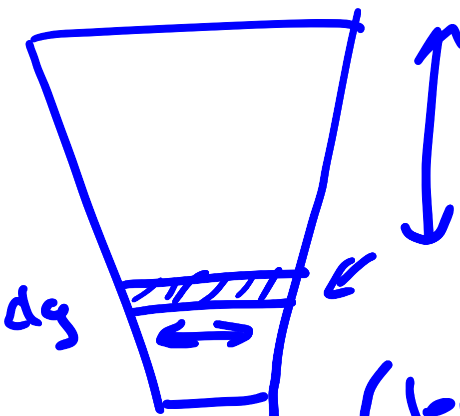
(Hint for a HW prob)

Hydrostatic force: This is total force acting on submerged object like a dam or submarine.

The force exerted at depth d is $F(d) = \rho A d$ where A is the area of the object at this depth and ρ is weight-density of the liquid.

Dam

dy

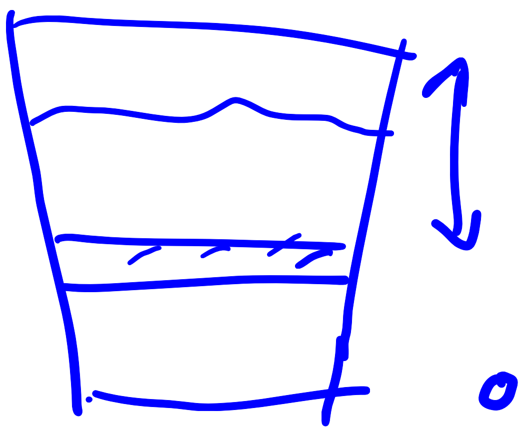


Area · depth = Press

(length dy) · depth

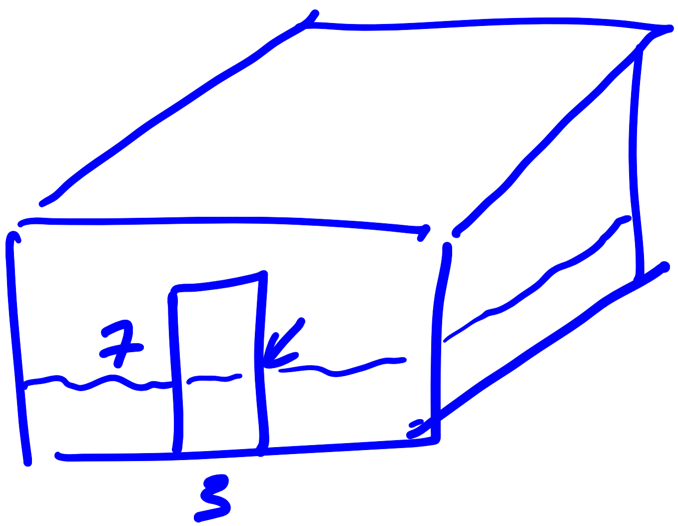
To find the total hydrostatic force on a submerged surface

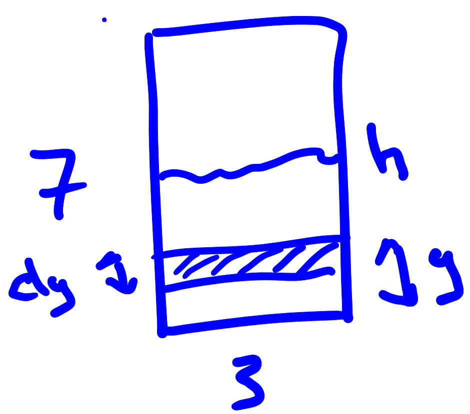
1. Sketch the object
2. Cut into horizontal slices, compute length of the cross section.
3. Multiply this by the depth.
4. Integrate between bottom and top levels.
5. Multiply by weight-density of the liquid $62.4 \ \text{lb}/\text{ft}^3$



A room is flooding with water that is held back by a rectangular door that is 3 feet wide and 7 feet high. The door can withstand 2000 lbs of force before breaking. How high is the water when this happens?

The weight density of water is 62.4 lb/ft^3.





$h-y$

$A = 3 \cdot dy$

$$\int_0^h \underbrace{(h-y)}_{\text{depth}} \underbrace{3\,dy}_{\text{area}}$$

$$= 3\left[hy - \frac{1}{2}y^2\right]_0^h$$

$$= 3\left[h^2 - \frac{1}{2}h^2\right]$$

$$= \frac{3}{2}h^2 \cdot (62.4)$$

$$\frac{3}{2}h^2(62.4) = 2000$$

$$h = \sqrt{\frac{2000}{62.4}\,\frac{2}{3}} \approx 4.6$$

Review for Quiz 6:

3 problems giving formulas: arclength, surface area for both rotations ✓

2 problems giving arclength integrals for specific functions.

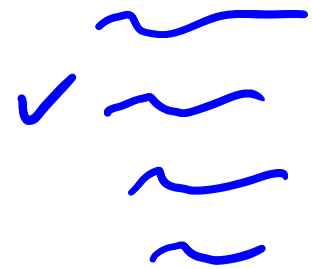
2 problems giving surface integrals for specific functions.

Compare size of two previous areas; which is larger or smaller.

Pumping water out of tank; set up integral

Compute work needed to bring repelling particles together

What is formula for arclength of graph of f on $[a, b]$?



$$\int_a^b \sqrt{1 + |f'(x)|^2}\, dx$$

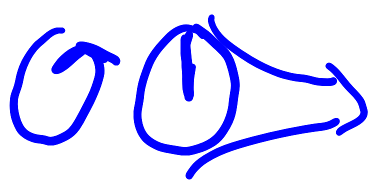
What is formula for area of f on $[a, b]$ after rotating around x-axis?

$$\int_a^b \underbrace{2\pi f(x)} \sqrt{1 + |f'(x)|^2}\, dx$$

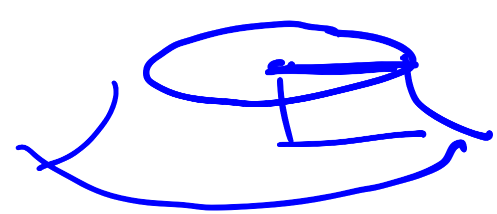
What is formula for area of f on $[a, b]$ after rotating around y-axis?

$$\int_a^b \underbrace{2\pi x} \sqrt{1 + |f'(x)|^2}\, dx$$

x-rot

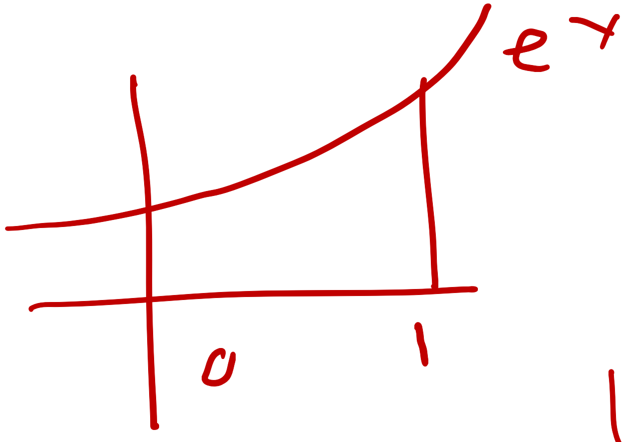


$2\pi r = \underbrace{2\pi f(x)}$



$2\pi r = \underbrace{2\pi x}$

What is arclength of e^x on $[0, 1]$?



$f = e^x \quad f' = e^x$

$(e^x)^2 = e^{2x}$

$$\int_0^1 \sqrt{1 + e^{2x}}\, dx$$

What is arclength of $\sin^2(x)$ on $[0, \pi]$?

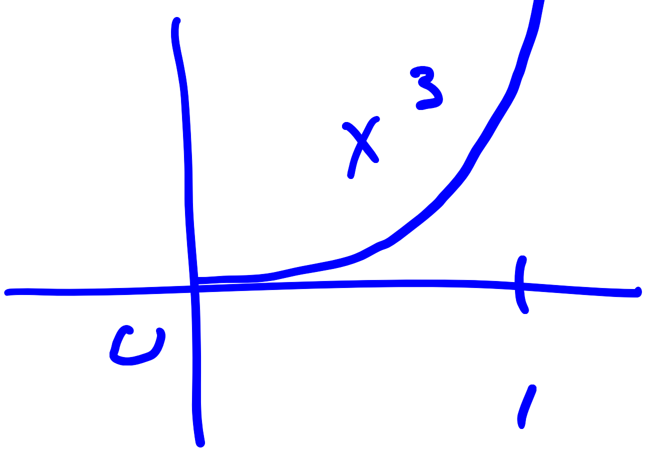
$f = \sin^2 \quad f' = 2\sin \cdot \cos$

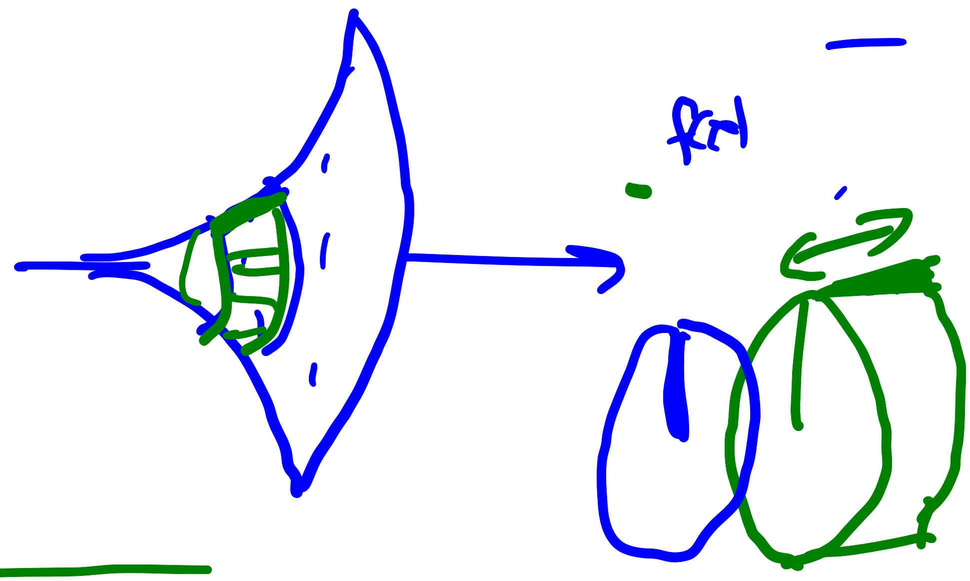
$$\int_0^{\pi} \sqrt{1 + (2\sin x \cos x)^2}\, dx$$

$$= \int_0^{\pi} \sqrt{1 + 4\sin^2 x \cos^2 x}\, dx$$

$$(x^3)' = 3x^2$$

What is surface area of x^3 on $[0,1]$ rotated around x-axis?





$$\int_a^b 2\pi f(x)\cdot\sqrt{1+(f')^2}\,dx$$

$$= \int_0^1 2\pi x^3\sqrt{1+(3x^2)^2}\,dx$$

$$= 2\pi\int_0^1 x^3\sqrt{\underbrace{1+9x^4}}\,dx$$

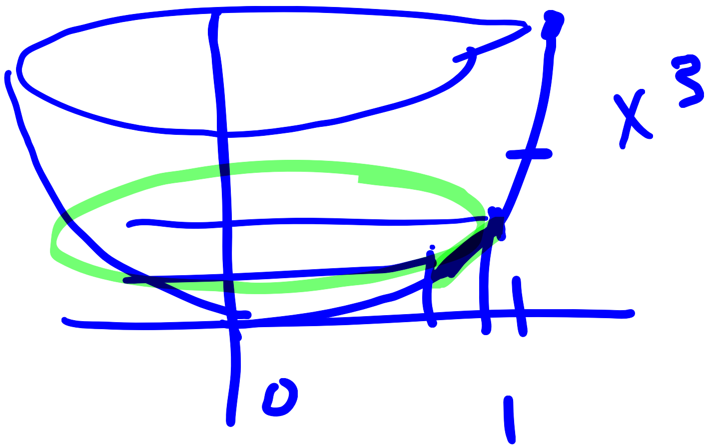
$u = 1+9x^4$

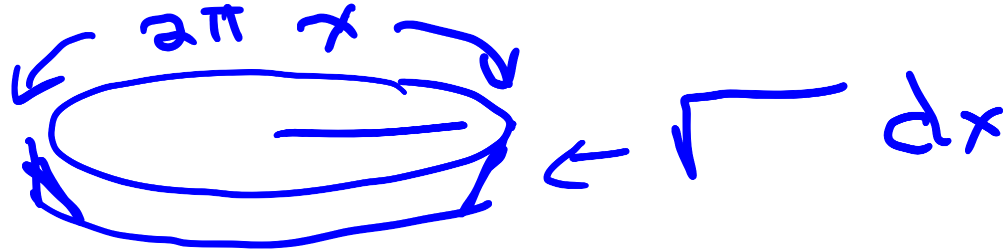
$du = 36x^3$

. . .

What is surface area of x^3 on $[0,1]$ rotated around y-axis?

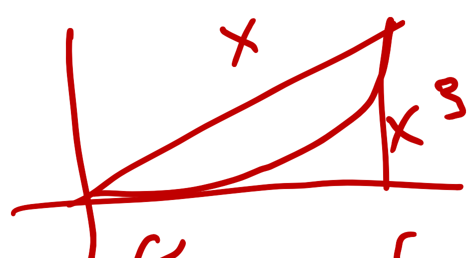
$\cancel{2\pi f(x)}$



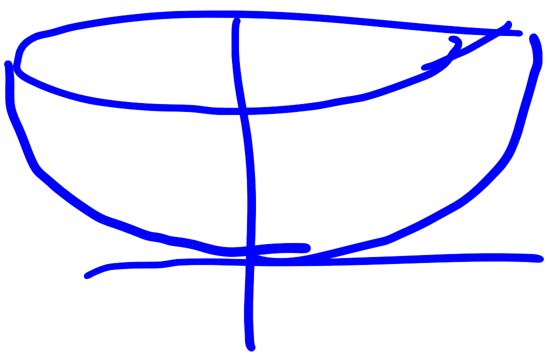


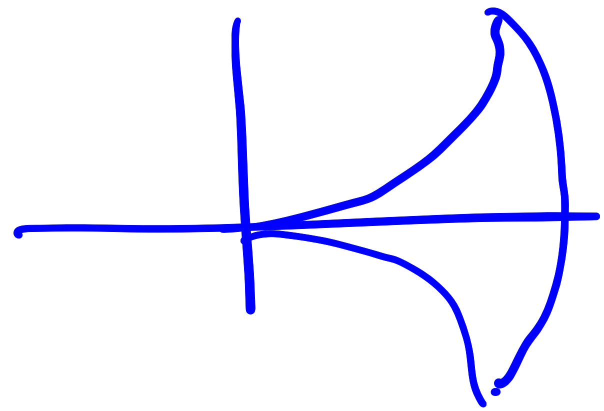
$$\int_0^1 2\pi x \cdot \sqrt{1+(3x^2)^2}\,dx \qquad y\text{ rot}$$

$$\int_0^1 2\pi x^3 \sqrt{1+(3x^2)^2}\,dy \qquad x\text{-rot}$$



Which is larger: the area of x^3 rotated around x-axis or y-axis?

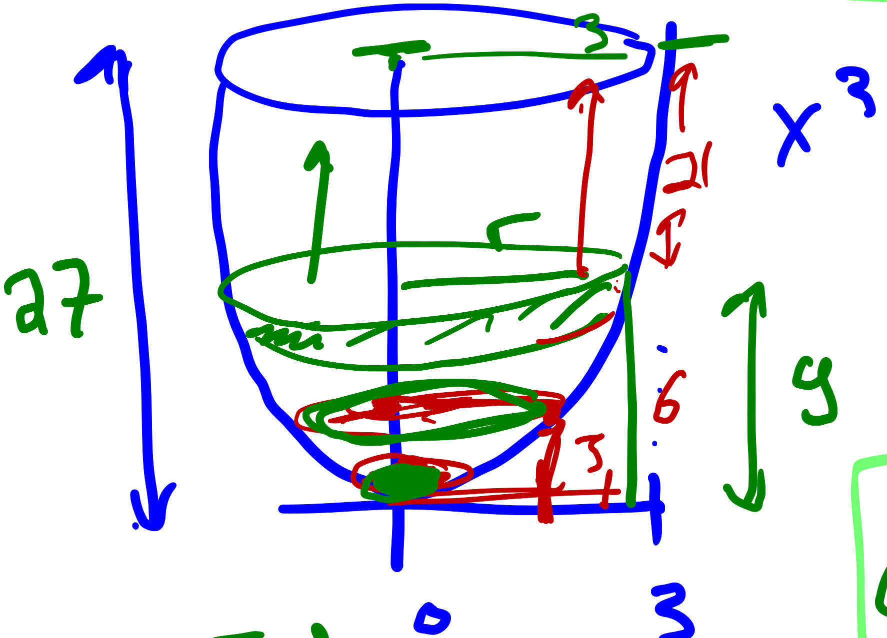




which area is larger?

Y rot is bigger because $x > x^3$ on $[0,1]$

A water tank is shaped like x^3 on $[0,3]$ rotated around the y-axis. There are 6 feet of water in the tank. The work is needed to pump the water over the top of the tank is 62.4 lb/ft^3 times what integral?



$r = y^{1/3}$

$A = \pi r^2$

$= \pi y^{2/3}$

$$\boxed{62.4 \cdot \int_0^6 \underbrace{\pi y^{2/3}}_{\text{weight}} \cdot \underbrace{(27-y)}_{\text{dist}} \, dy}$$

vol

distance

$W = F \cdot d$

$y = x^3$

$x = y^{1/3}$

$r = y^{1/3}$

$A = \pi r^2 = \pi y^{2/3}$

$$= (62.4)\pi \int_0^6 27y^{2/3} - y^{5/3}$$

$$= (62.4)\pi \left[27\tfrac{3}{5} y^{5/3} - \tfrac{5}{8} y^{8/3}\right]_0^6$$

$$= (62.4)\pi \left[27 \cdot \tfrac{3}{5} 6^{5/3} - \tfrac{5}{8} 6^{8/3}\right]$$

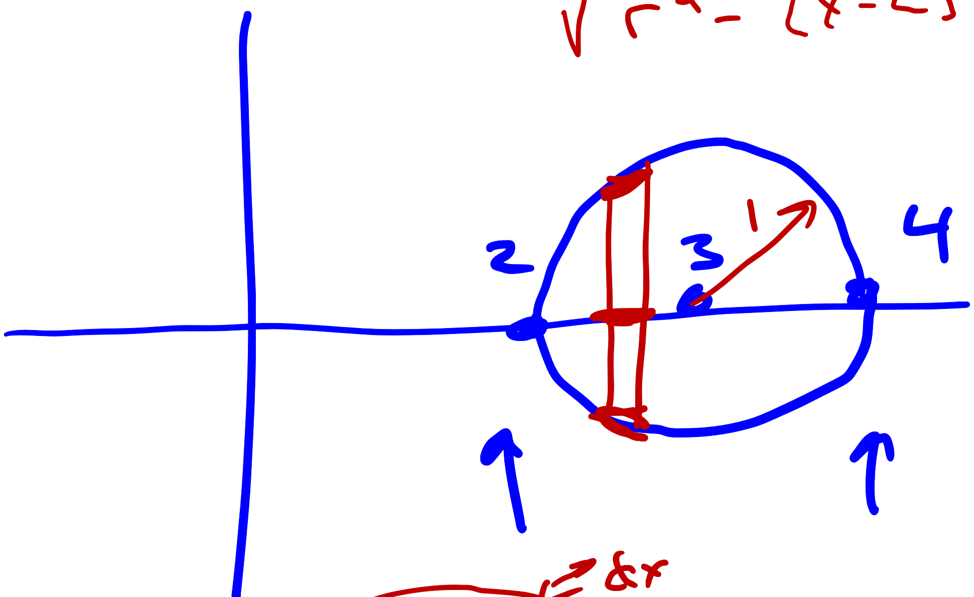
$+0$

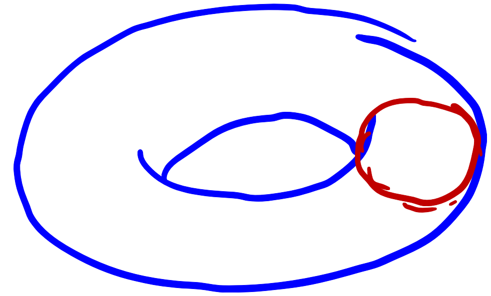
Hooke's law says that the force needed to compress or stretch a spring x units from equilibrium is $F(x) = kx$, for some constant k depending on the spring. Suppose it takes a force of 8 lb to stretch a spring 6 inches. How much work in foot-lbs is done to stretch the spring 1 foot (= 12 inches)?

Oct 8 Office Hours

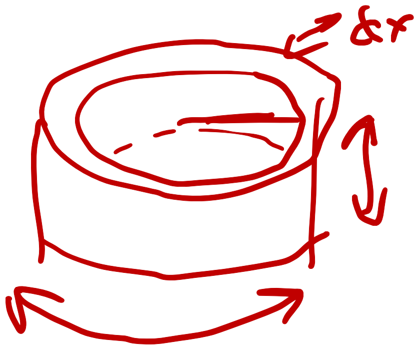
$\sqrt{r^2-(x-c)^2}$

$\sqrt{1-(x-3)^2}$





"Torus"



$2\pi x$

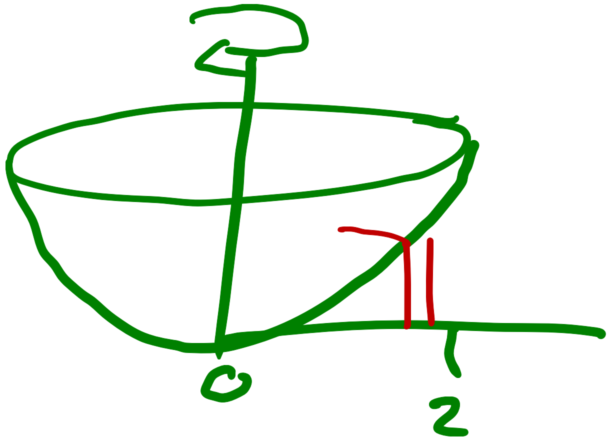
Vol $\approx$ height $\times$ length $\times$ width

$$2\sqrt{1-(x-3)^2} \times \underline{2\pi x} \times dx$$

$$\int_2^4 2\pi x \, 2\sqrt{1-(x-3)^2} \, dx$$

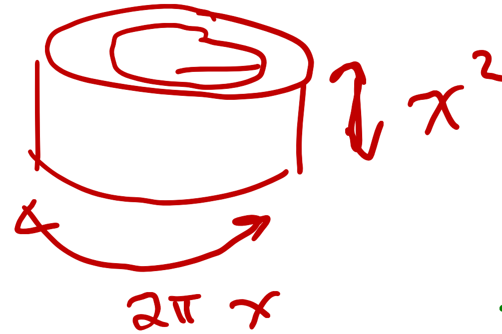
$$= 4\pi \int_2^4 \underline{x} \sqrt{\underbrace{1-(x-3)^2}} \, dx$$

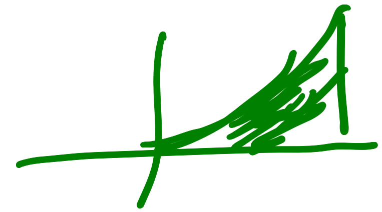
$du = -2x\,dx$ $\quad u = 1-(x-3)^2$



x^2

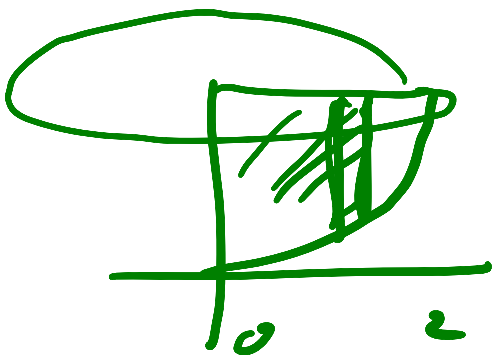
Volume





$$\int_0^2 2\pi x\,(x^2)\,dx$$

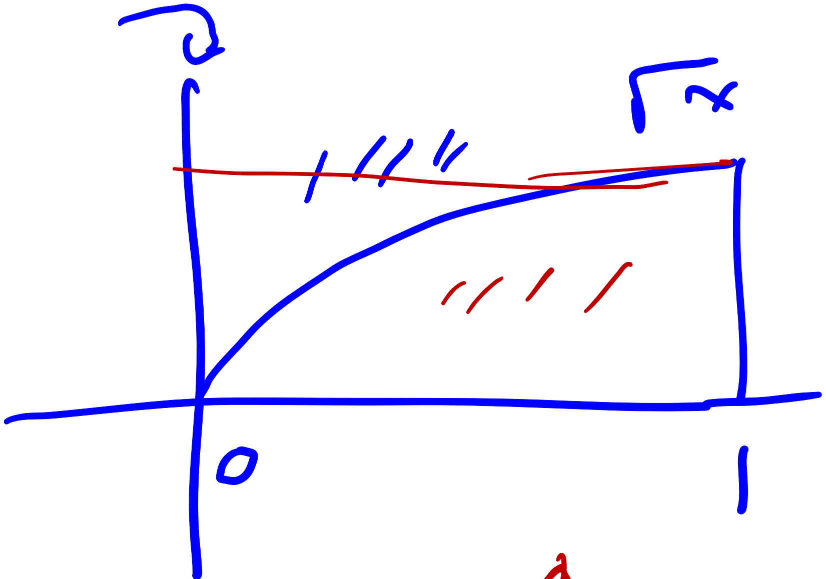
$$= 2\pi \int_0^2 x^3\,dx$$



4

x^2

$$\int_0^2 2\pi x\left[4 - x^2\right]dx$$



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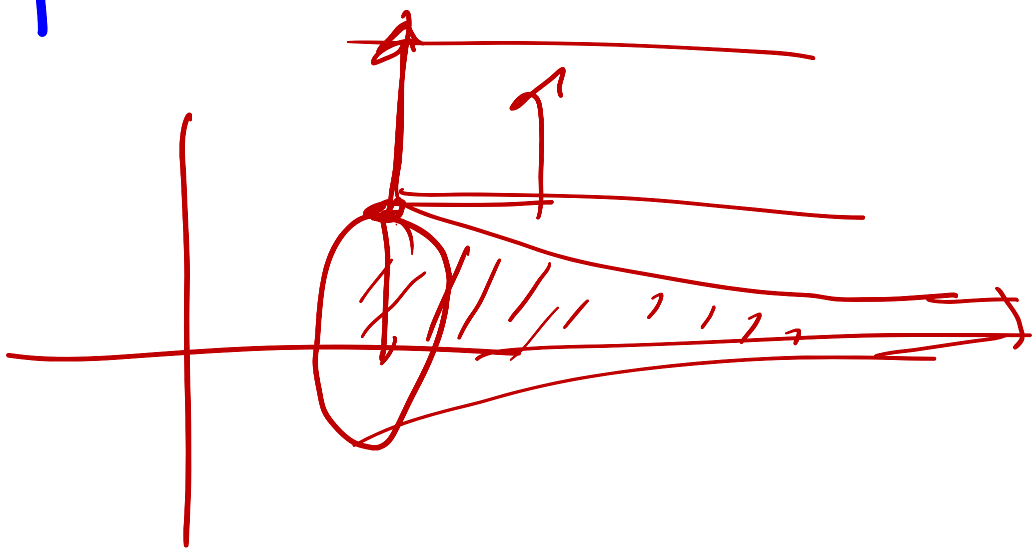
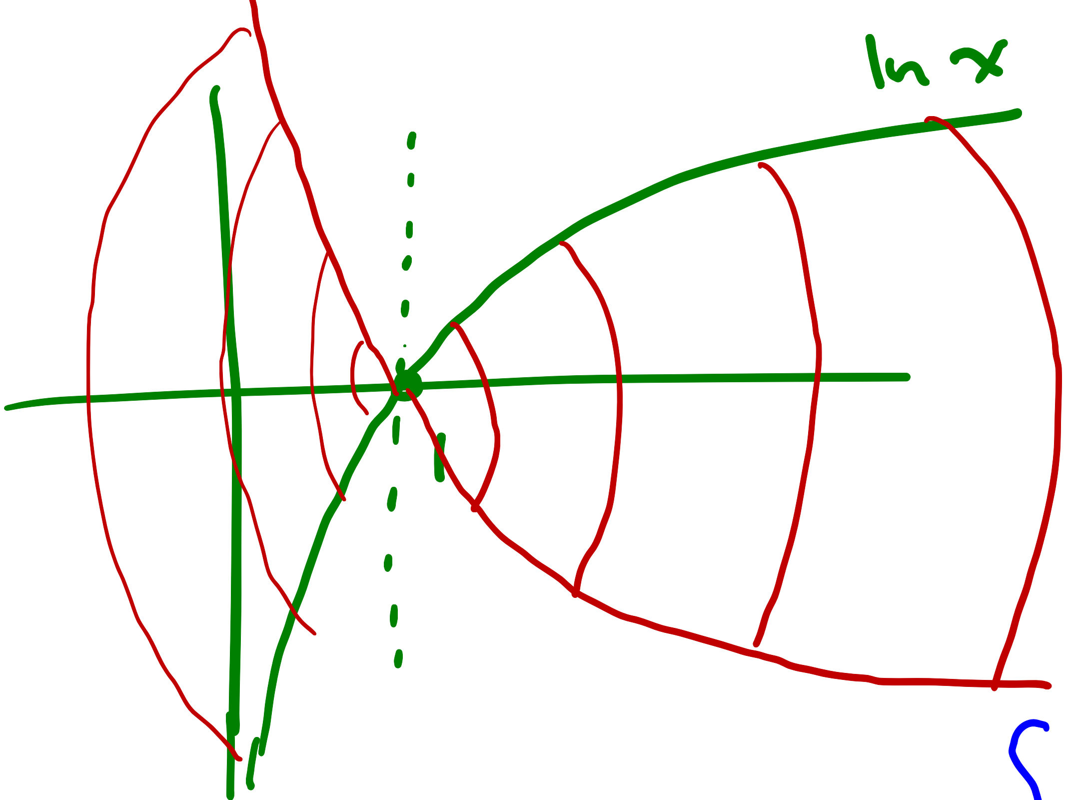
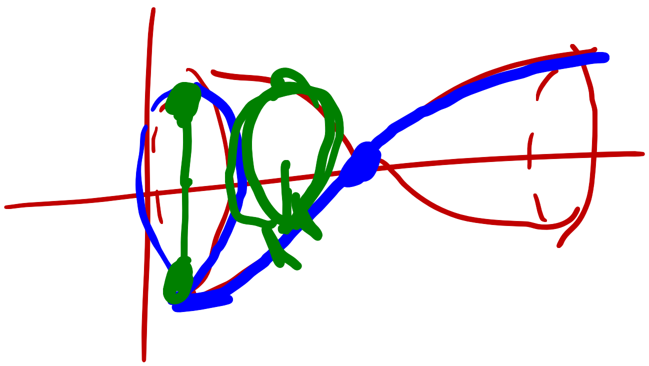
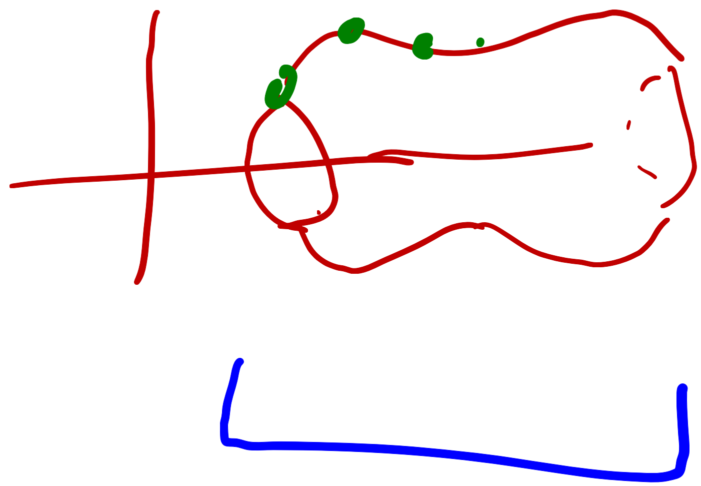




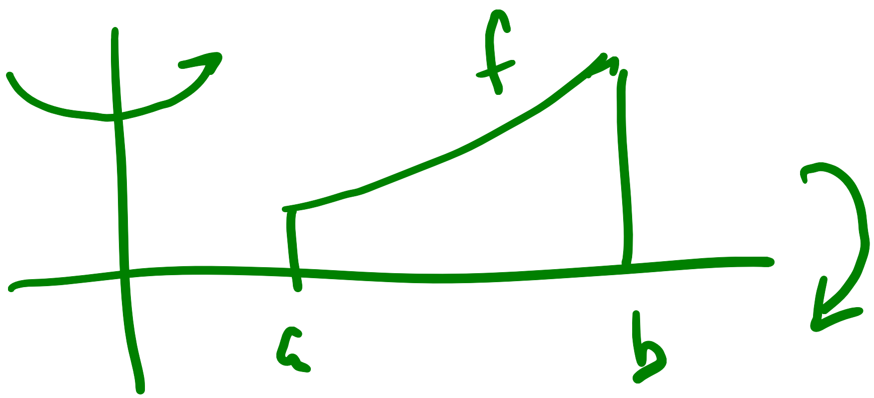
$$\int 2\pi\, |f(x)|^2$$

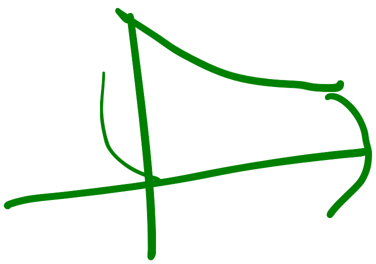
$$\int 2\pi\, |f(x)| \sqrt{1 + |f'(x)|}$$

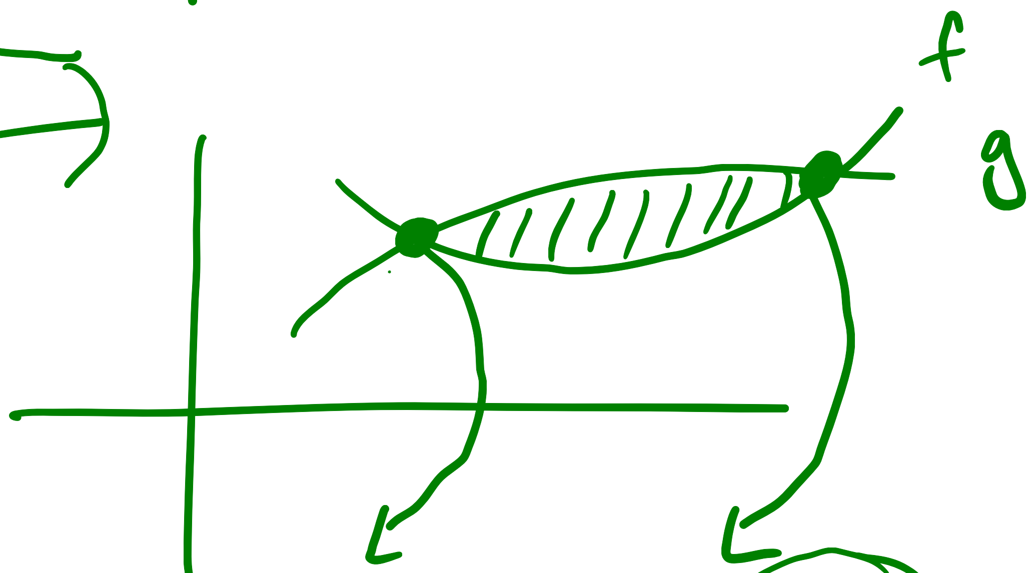




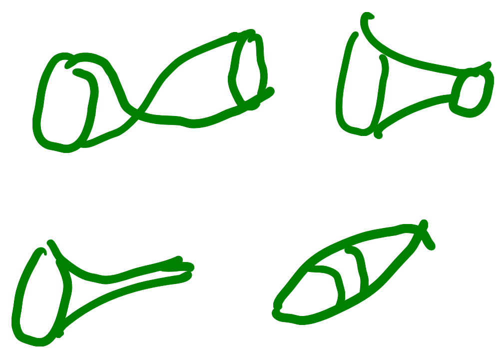
f, a, b, which axis



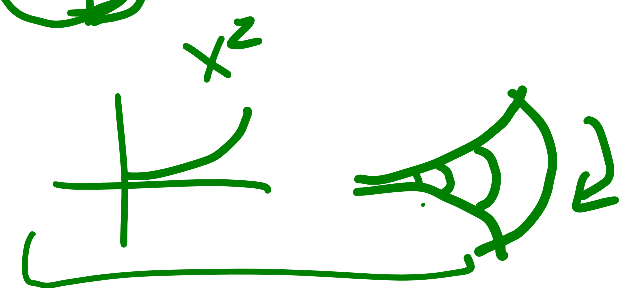


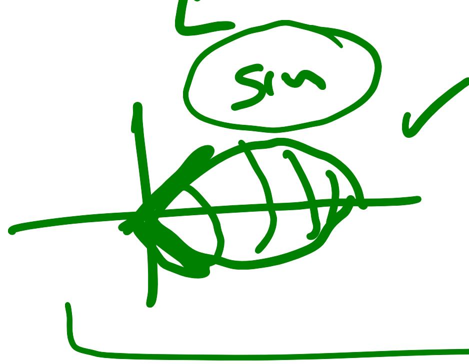


sin x on [0, π]

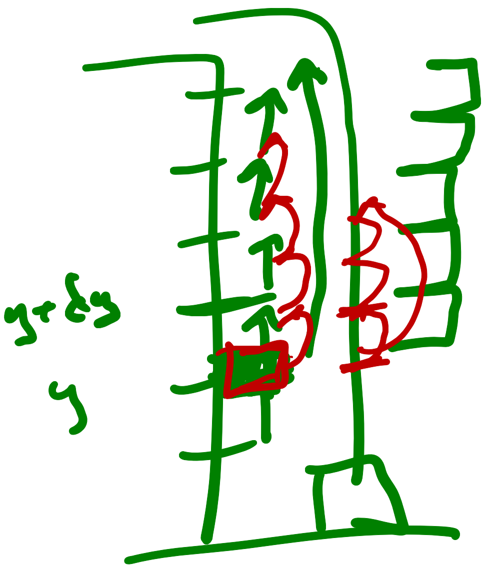


$\frac{1}{x}$ [2, 3]



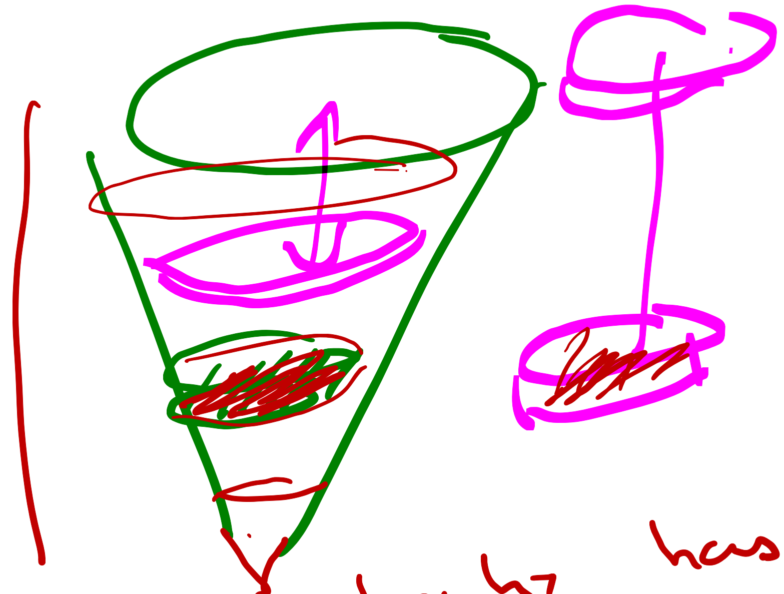


Bucket versus Water Tank

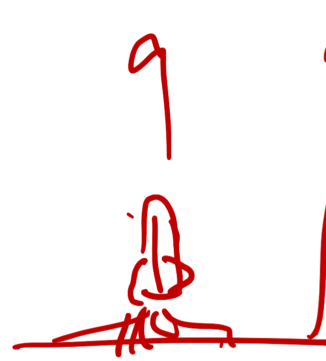


$$\int \left(\begin{array}{l}\text{weight at}\\ \text{height } y\end{array}\right) \cdot dy$$

$$\boxed{W = F \cdot d}$$ constant Force F



each height has different weight & distance.



1000 miles