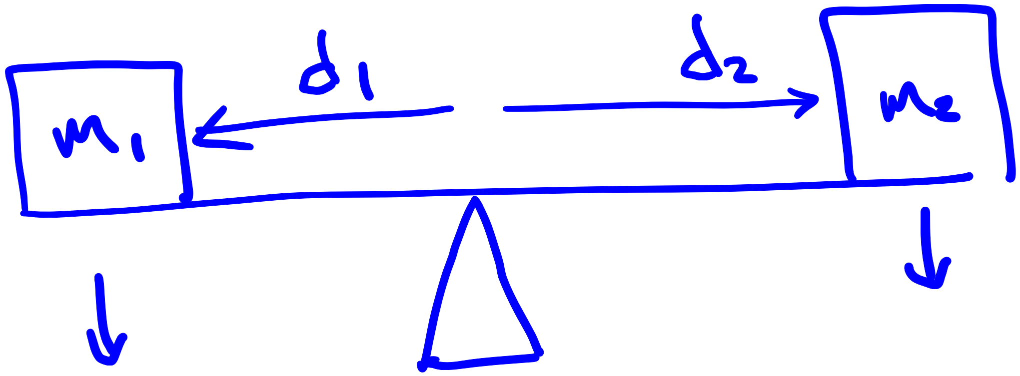
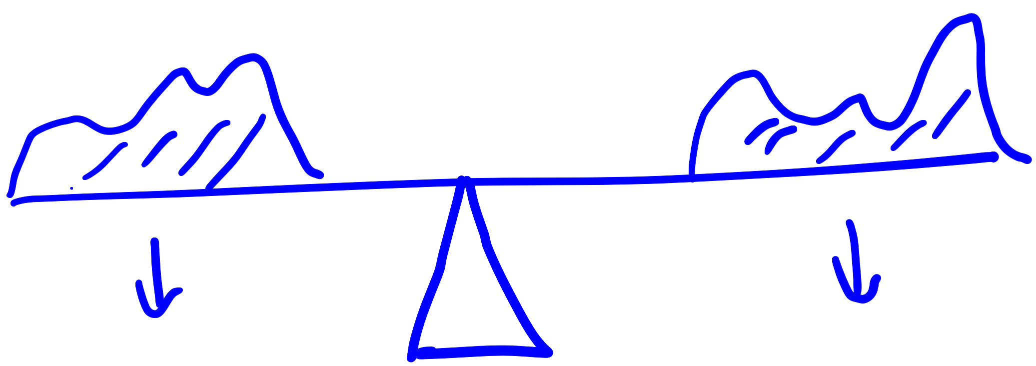
MAT 126.01, Prof. Bishop, Tuesday, Oct 13, 2020
Section 2.6 Moments and Centers of Mass
Theorem of Pappus

See-Saw example: $m_1 d_1 = m_2 d_2$.



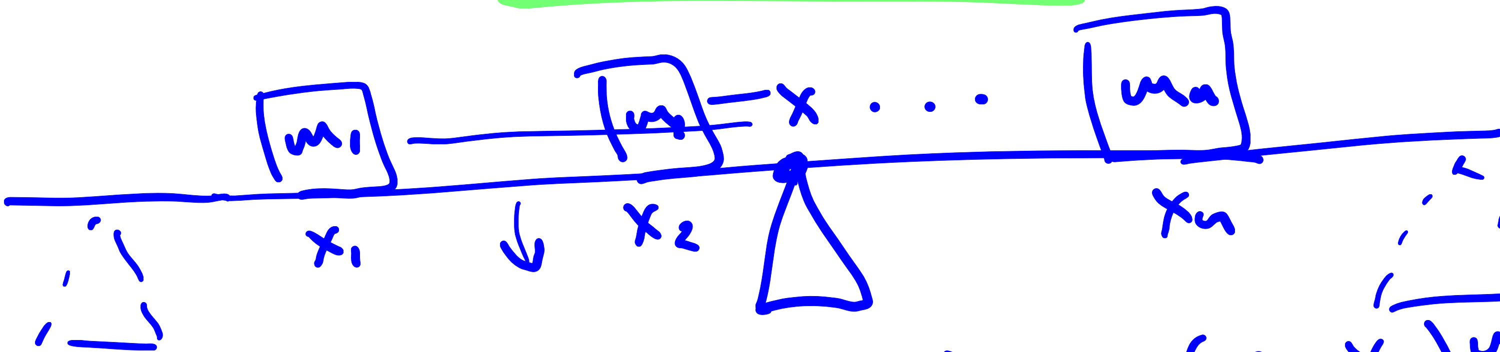
if $m_1 d_1 > m_2 d_2$

then left goes down



Center of weight of a finite number of masses on line:

$$\boxed{\overline{x} = \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k}} = \frac{M}{m}$$



$$\underbrace{m_1(x-x_1)+m_2(x-x_2)}_{\text{left of } x}+\cdots(x-x_k)m_k$$

$$= (x_{k+1}-x)m_{k+1}+(x_{n-1}-x)m_{n-1}+(x_n-x)m_n$$

$$m_1(x-x_1)+\cdots+(x-x_n)m_n = 0$$

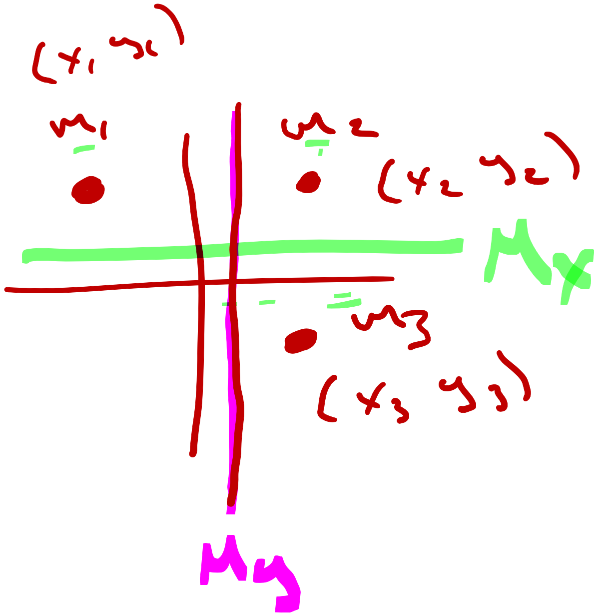
$$\sum_{k=1}^{n} m_k(x-x_k)=0$$

$$\sum m_k x - \sum m_k x_k = 0$$

$$x\sum m_k = \sum m_k x_k$$

$$\bar{x} = \frac{\sum m_k x_k}{\sum m_k} = \frac{\text{moment}}{\text{mass}}$$

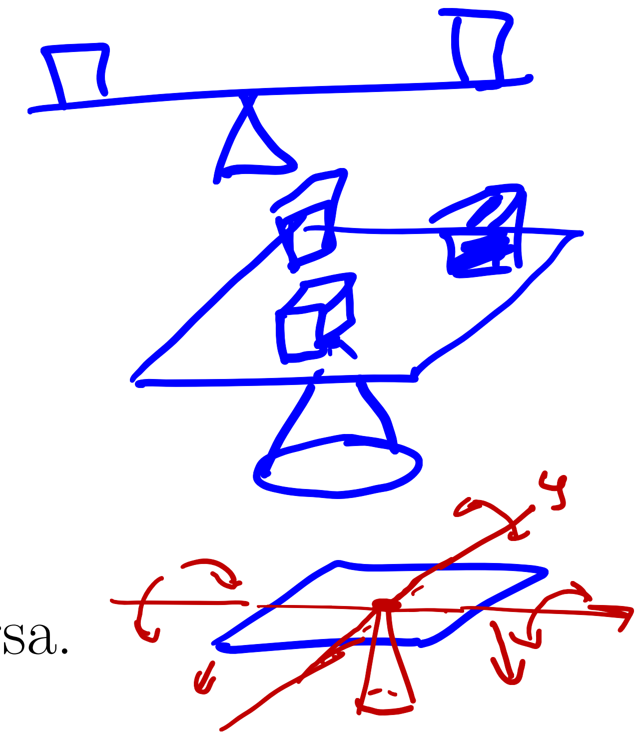
Moments with respect to x-axis and y-axis :



$$M_y = \sum_{k=1}^{n} m_k x_k$$

$$M_x = \sum_{k=1}^{n} m_k y_k$$

$$\bar{x} = \frac{M_y}{m}, \qquad \bar{y} = \frac{M_x}{m}$$



Moment w.r.t. x-axis is on the y-axis and vice versa.

(with respect to)

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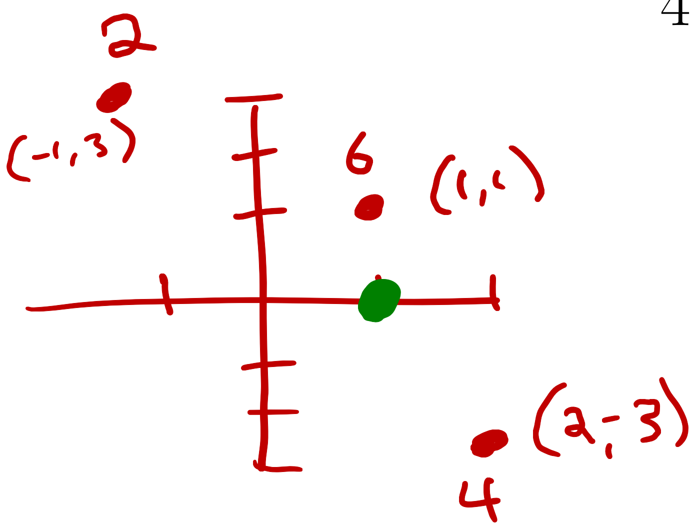
Find center of mass of

$$2 \text{ kg} \quad \text{at } (-1,3),$$
$$6 \text{ kg} \quad \text{at } (1,1),$$
$$4 \text{ kg} \quad \text{at } (2,-3),$$



$$M_y = 2(-1) + 6(1) + 4(2)$$
$$= -2 + 6 + 8 = 12$$

$$M_x = 2(3) + 6(1) + 4(-3)$$
$$= 6 + 6 - 12 = 0$$

$$m = 2 + 6 + 4 = 12$$

$$\bar{x} = \frac{M_y}{m} = \frac{12}{12} = 1$$
$$\bar{y} = \frac{M_x}{m} = \frac{0}{12} = 0$$

lamina = thin plate represented by region in the plane, constant density.

centroid = center of mass.

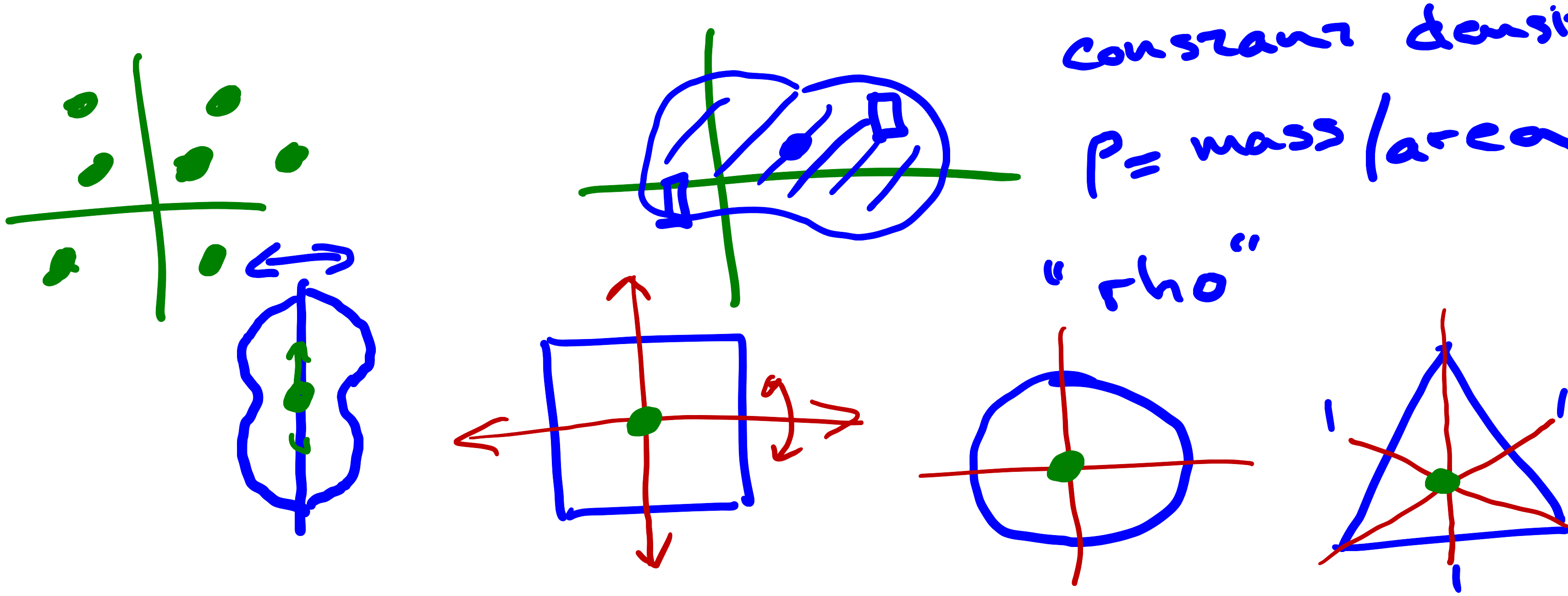
Symmetry Principle: if a region is symmetric with respec to a line, then the centroid in on that line

Corollary: if there are two lines of symmetry the centroid is at the intersection point.

constant density

ρ = mass/area

"rho"



Suppose $R = \{(x, y) : a \leq x \leq b, 0 \leq y \leq f(x)\}$. Let ρ be the (constant) density of the associated lamina.

"rho"

$$\text{mass of lamina } = \rho \cdot \text{area}(R) = \rho \cdot \int_a^b f(x)dx$$

$\frac{\text{mass}}{\text{area}}$

$$\boxed{M_x = \rho \cdot \int_a^b \frac{1}{2}|f(x)|^2 dx}$$

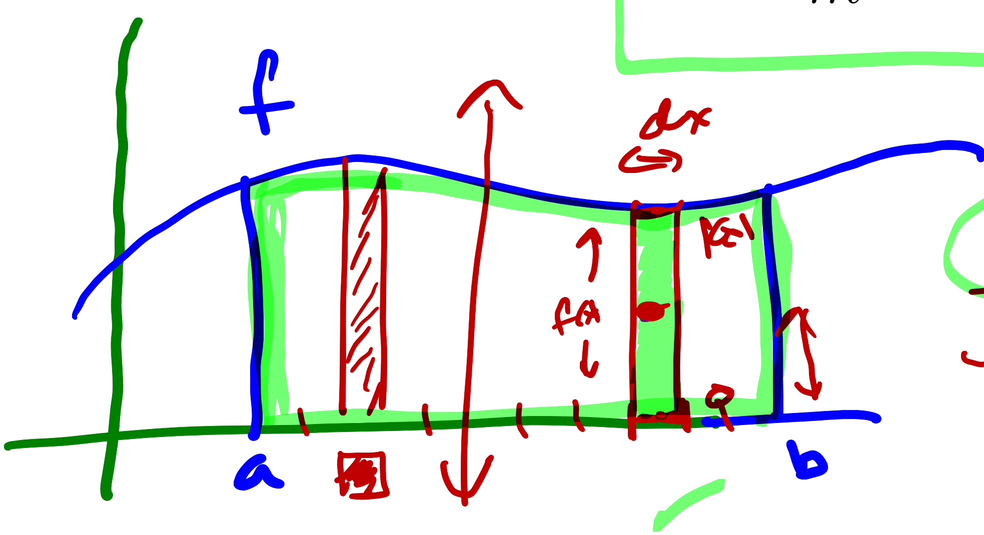
$= \int \left(\frac{1}{2} f\right) \cdot f \, dx$

$\rho = 1$

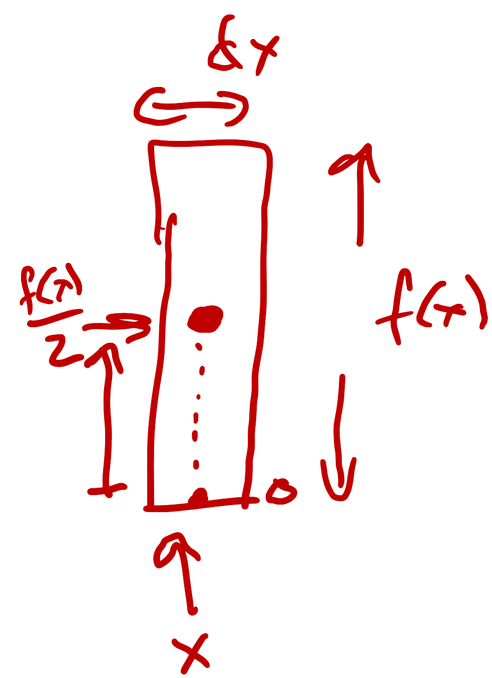
$$\boxed{M_y = \rho \cdot \int_a^b x|f(x)|dx}$$

$= \int (x) \cdot f \, dx$

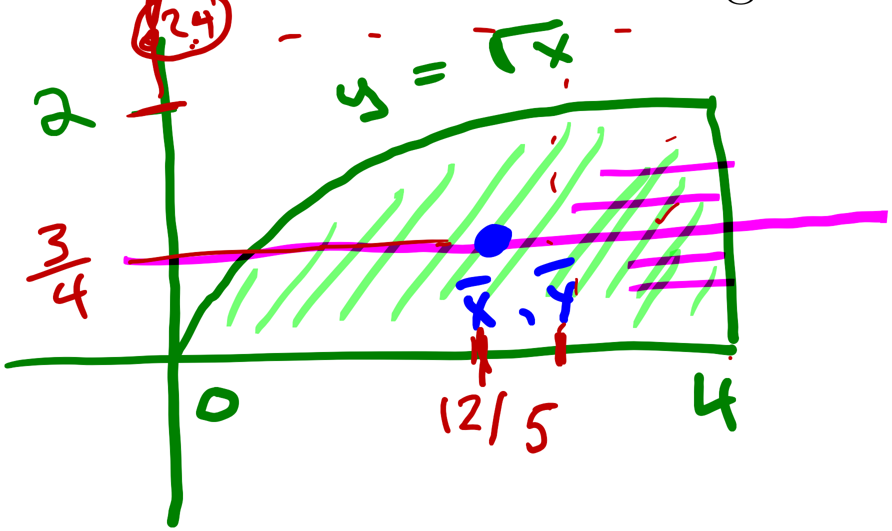
$$\boxed{\bar{x} = \frac{M_y}{m},} \quad \boxed{\bar{y} = \frac{M_x}{m}}$$



$\underbrace{\frac{f(x)}{2}}_{y}$ $\quad$ $\underbrace{f(x)\,dx}_{\text{mass}}$



Find the center of mass of region below $y = \sqrt{x}$ and above $[0, 4]$.



$$\bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}$$

$$\rho = 1$$

$$m = \int_0^4 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_0^4 = \frac{2}{3}(4)^{3/2} = \frac{16}{3}$$

$$M_x = \int_0^4 \frac{f(x)^2}{2}\,dx = \int_0^4 \frac{x}{2} = \frac{1}{4}x^2\Big|_0^4 = 4$$

$$M_y = \int_0^4 x f(x)\,dx = \int_0^4 x^{3/2}\,dx = \frac{2}{5}x^{5/2}\Big|_0^4 = \frac{64}{5}$$

$$\bar{y} = \frac{M_x}{m} = \frac{4}{16/3} = \frac{3}{4} \qquad \bar{x} = \frac{M_y}{m} = \frac{64/5}{16/3} = \frac{4\cdot 3}{5} = 12/5 \approx 2.4$$

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Suppose $R = \{(x, y) : a \le x \le b, g(x) \le y \le f(x)\}$. Let ρ be the (constant) density of the associated lamina.

$$\text{mass of lamina } = \rho \cdot \text{area}(R) = \rho \cdot \int_a^b f(x) - g(x)dx$$

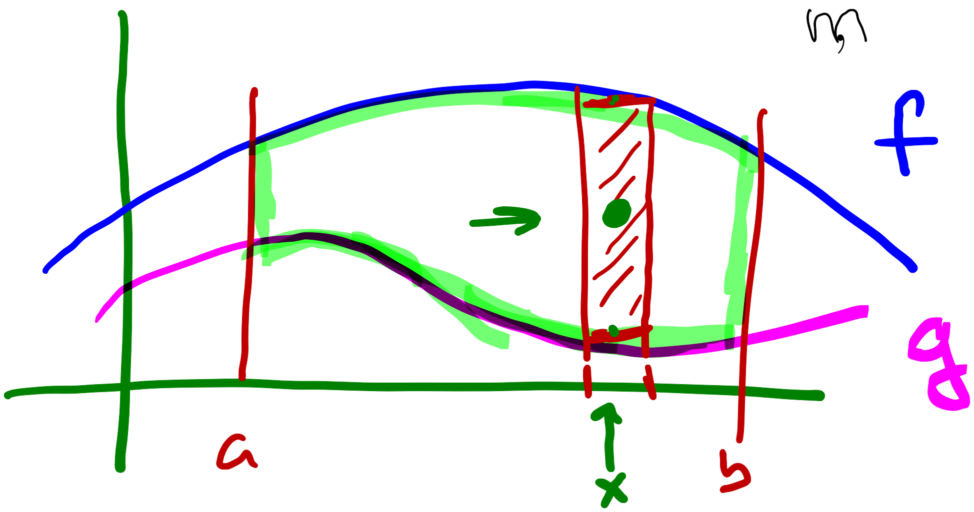
$$M_x = \rho \cdot \int_a^b \frac{1}{2}(|f(x)|^2 - |g(x)|^2)dx$$

$(a+b)(a-b)$

$a^2 - b^2$

$$M_y = \rho \cdot \int_a^b x(f(x) - g(x))dx$$

$$\overline{x} = \frac{M_y}{m} \qquad \overline{y} = \frac{M_x}{m}$$

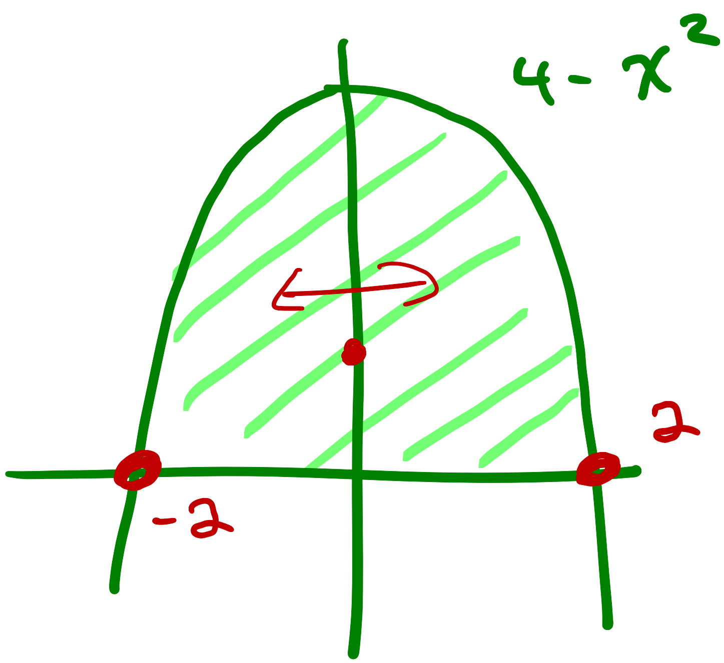


$\underbrace{\frac{f(x)+g(x)}{2}}_{\text{y-coor}}$

$\underbrace{(f(x)-g(x))dx}_{\text{mass}}$

x

Find the center of mass of the region bounded above by $y = 4 - x^2$ and below by $y = 0$. (Use symmetry).



$$4 - x^2 = 0$$
$$4 = x^2$$
$$\pm 2 = x$$

$$m = \int_{-2}^{2} (4 - x^2) = \left(4x - \tfrac{1}{3}x^3\right)_{-2}^{2} = 16 - \frac{16}{3} = 32/3$$

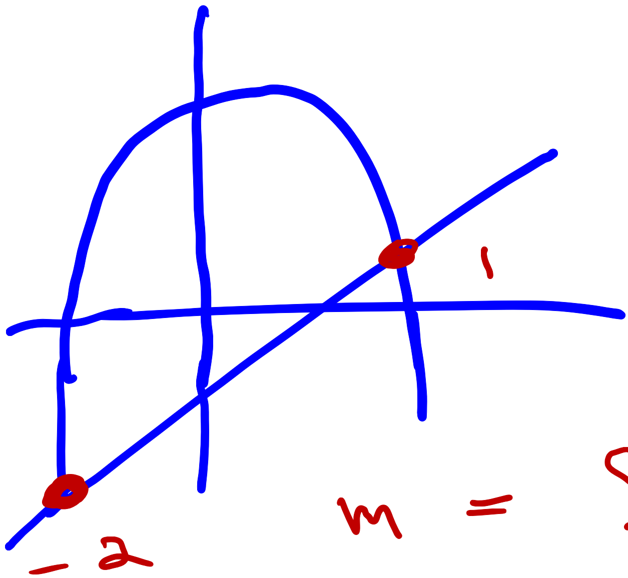
$$M_x = \int_{-2}^{2} \frac{1}{2}(4 - x^2)^2 = \frac{1}{2}\int_{-2}^{2} (16 - 8x^2 + x^4)$$

$$= \frac{1}{2}\left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_{-2}^{2}$$

$$= \frac{1}{2}\left[64 - \frac{128}{3} + \frac{64}{5}\right]$$

$$M_y = 0$$

Find the center of mass of the region bounded above by $y = 1 - x^2$ and below by $y = x - 1$.



$$1 - x^2 = x - 1$$

$$0 = x^2 + x - 2$$

$$= (x+2)(x-1)$$

$$x = -2 \qquad x = 1$$

$$m = \int_{-2}^{1} (1-x^2) - (x-1)\,dx$$

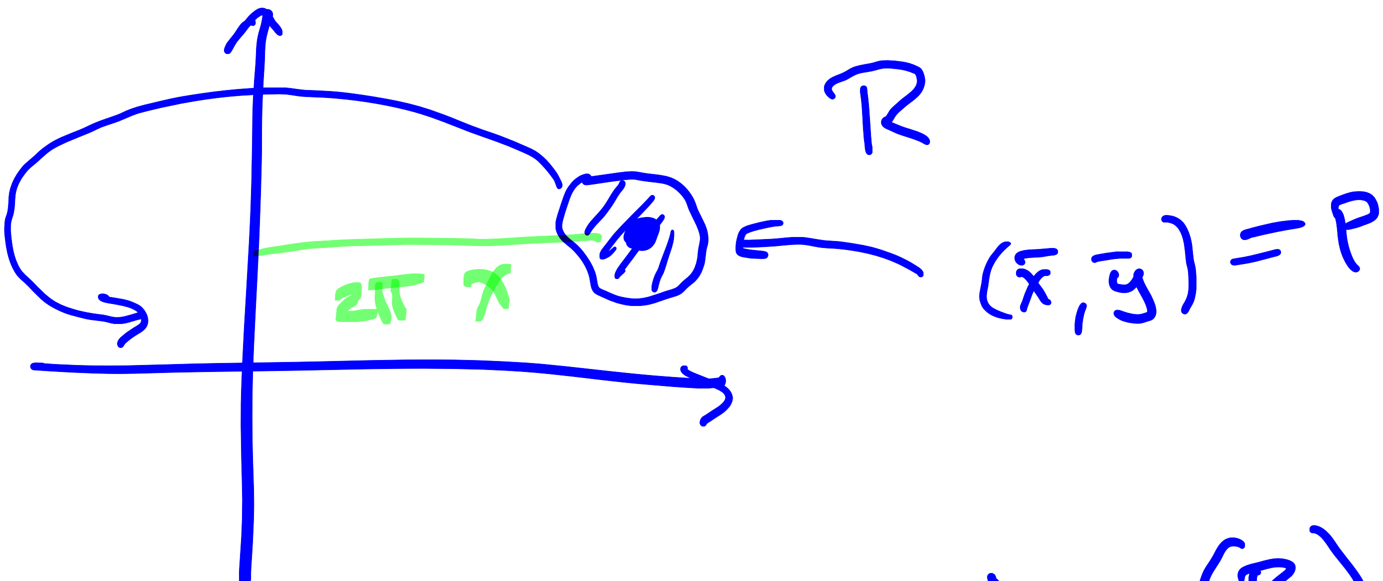
$$M_x = \int_{-2}^{1} \frac{1}{2}(f^2 - g^2) = \int_{-2}^{1} \frac{1}{2}\left((1-x^2)^2 - (x-1)^2\right)dx$$

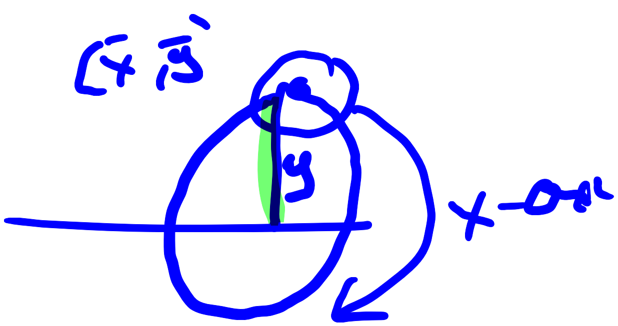
$$M_y = \int_{-2}^{1} x(f-g) = \int_{-2}^{1} x\left((1-x^2) - (x-1)\right)dx$$

$$\vdots$$

$$\bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}$$

Theorem of Pappus: If R is a planar region and L a line not hitting L then the volume formed by rotating R around L is the area of R multiplied by the distance traveled by the center of mass around L.





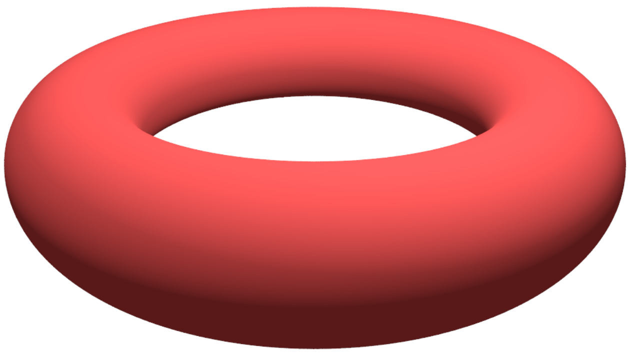
$$\text{Vol} = \underbrace{\text{Area}(R)}\cdot \text{distance } P \text{ travels}$$

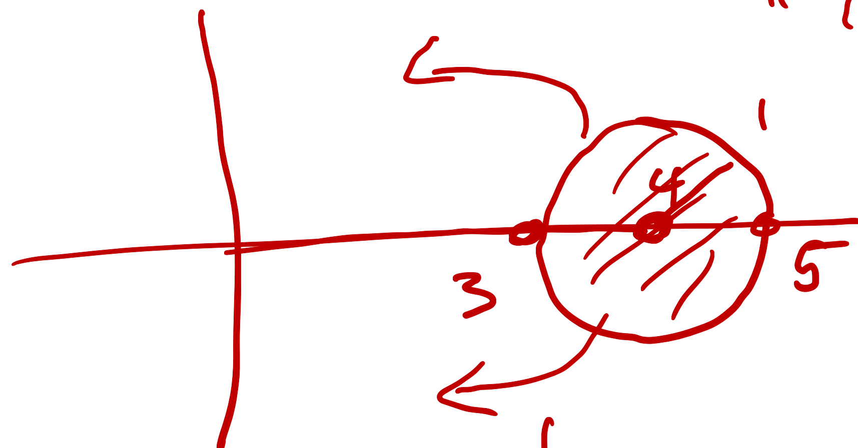
$$= A(R) \cdot 2\pi\,\bar{x} \quad (y \text{ rotation})$$

$$= A \cdot 2\pi\,\bar{y} \quad (x \text{ rotation})$$

Let R be the circle of radius 1 centered at $(4, 0)$. What is the volume of the torus formed by rotating this around the y-axis?

"Torus"





$$\text{Vol} = \int_3^5 \cdots \sqrt{\cdots\cdots}$$

Area $= \pi r^2$

$= \pi \cdot 1$

$= \pi$

$$\text{Vol} = \text{area} \cdot \text{distance traveled}$$
$$= (\pi) \cdot (8\pi) = 8\pi^2$$

distance

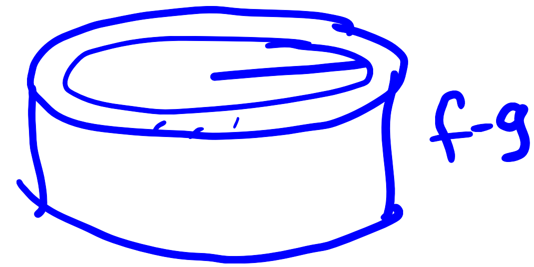
$= 2\pi \bar{x}$

$= 2\pi \cdot 4$

$= 8\pi$

Proof of Theorem of Pappus: By method of shells

$$V = 2\pi \int_a^b x(f(x) - g(x))dx.$$



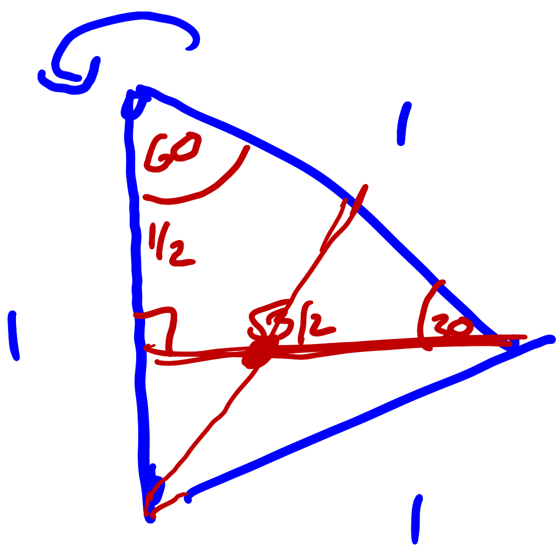
$$\text{Area} = m = \int_a^b (f(x) - g(x))dx$$

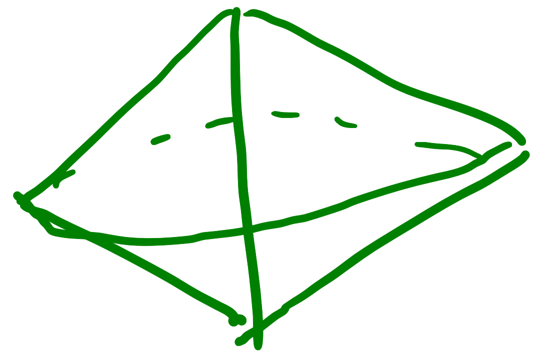
Distance traveled by center of mass

$$d = 2\pi\overline{x} = 2\pi\frac{M_y}{m} = \frac{2\pi\int_a^b x(f(x) - g(x))dx}{\int_a^b (f(x) - g(x))dx}$$

So $V = dA$.

An equilateral triangle of with side length one is rotated around one of its sides. What is the resulting volume?





$\bar{x}$

Area $= \left(\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)2 = \frac{\sqrt{3}}{4}$

MAT 126

Office Hours

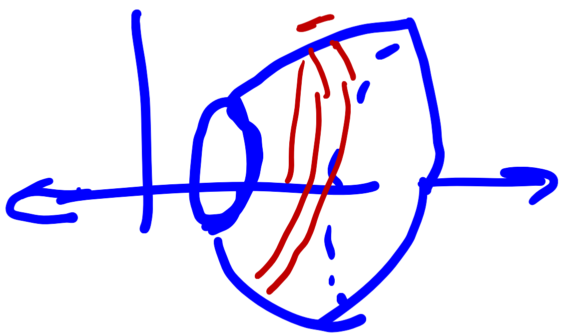
HW 7, Q 6

$f(x) = 6\sqrt{x} = 6x^{1/2}$

$f'(x) = \underbrace{3x^{-1/2}}$

$y = 6\sqrt{x} \qquad 2.3 \le x \le 4.2$

rotate around x-axis, Find surface area.



$$\int_a^b 2\pi f(x)\cdot\sqrt{1+[f'(x)]^2}\,dx$$

$$= \underline{2\pi} \int_{2.3}^{4.2} 6\sqrt{x}\,\sqrt{1+9/x}\,dx$$

$$= 12\pi \int \sqrt{x+9}$$

$u = x+9 \quad du = x$

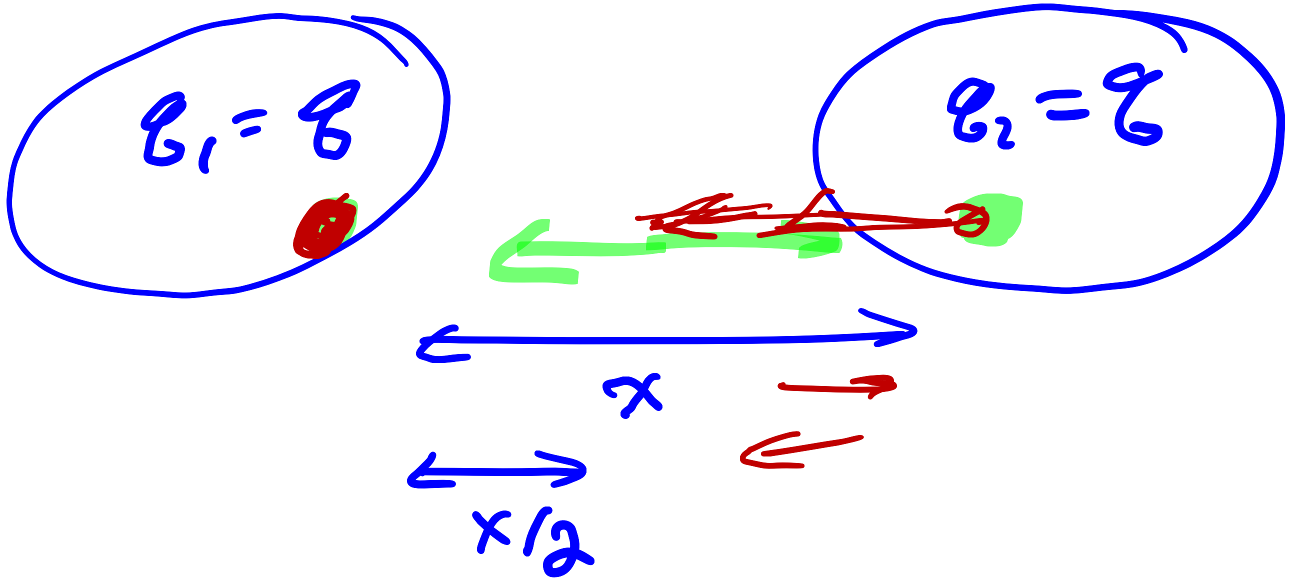
$$= 12\pi \int \sqrt{u}$$

$$= 12\pi\ \frac{2}{3}u^{3/2}$$

$$= 12 \times \frac{2}{3}(x+9)^{3/2}\Big|_{2.3}^{4.2}$$

$$2\pi 6\sqrt{x+\underbrace{6^2/4}_{9}}$$

$$\boxed{12\pi\sqrt{x+9}} = 2\pi 6\sqrt{x}\sqrt{1+9/x}$$



Quiz 6 Feb 10

$$\text{Force} = k \cdot q_1 q_2 \frac{1}{x^2}$$

$$\int_{x/2}^{x} F \cdot d \, dx = \int_{x/2}^{x} \frac{k q_1 q_2}{x^2} \, dx$$

$$= k q_1 q_2 \int \frac{1}{x^2} \, dx$$

$$= k q_1 q_2 \quad -\frac{1}{x} \Big|_{x/2}^{x}$$

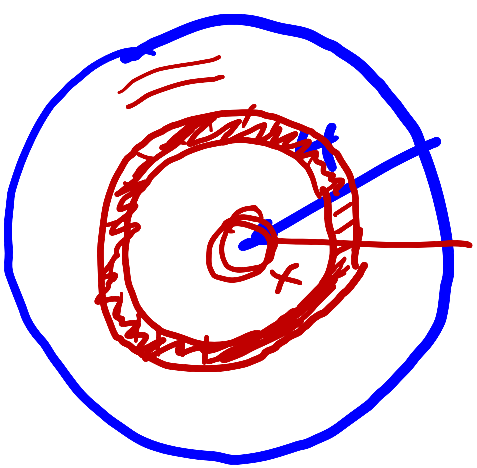
$$= k q_1 q_2 \left[-\frac{1}{x} - \left(-\frac{1}{x/2} \right) \right]$$

$$\boxed{\frac{2}{x} - \frac{1}{x}}$$

$$= k q_1 q_2 \cdot \frac{1}{x}$$

$$\boxed{= \frac{k q^2}{x}}$$

HW 7, Q 12



disk radius 4

density $\rho(x) = e^{-x}$

mass $= \underbrace{\int_0^4 e^{-x} \cdot 2\pi x \, dx}$

$= 2\pi \int_0^4 \underbrace{x e^{-x}} \, dx$

Area $= 2\pi x \, dx$

mass $\rho(x) \cdot 2\pi \cdot x \, dx$

$$(xe^{-x})' = e^{-x} + x(-e^{-x})$$

$$= e^{-x} - \underbrace{xe^{-x}}_{}$$

$$(-xe^{-x} - e^{-x})' = -\cancel{e^{-x}} + xe^{-x} + \cancel{e^{-x}}$$

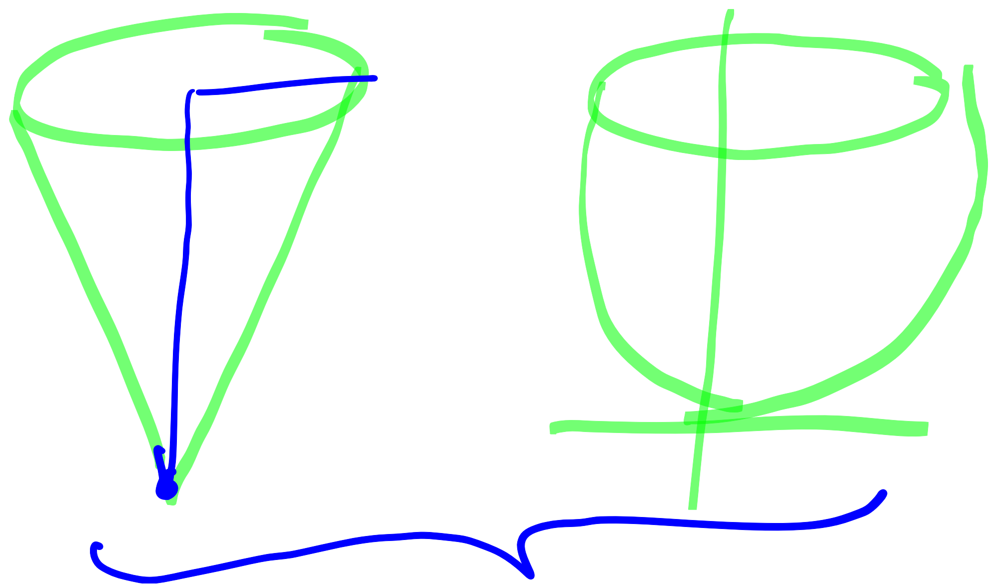
$$= xe^{-x}$$

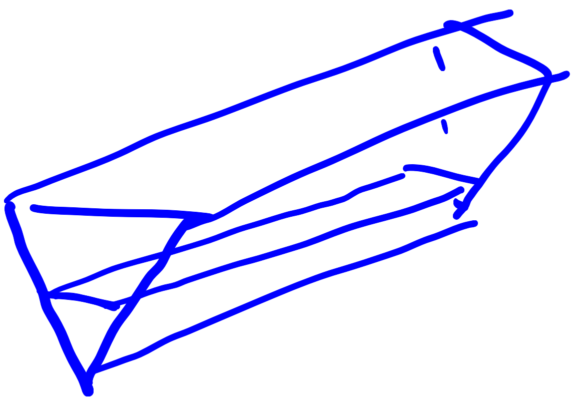
$$= 2\pi \left(-xe^{-x} - e^{-x}\right)\Big|_0^4$$

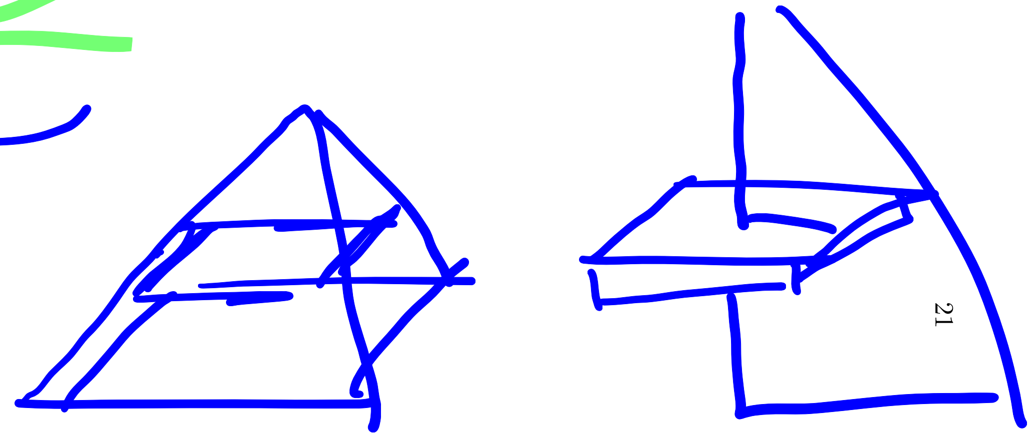
$$= 2\pi \left[\left(-4e^{-4} - e^{-4}\right) - \underbrace{\left(0 - e^{0}\right)}_{-1}\right]$$

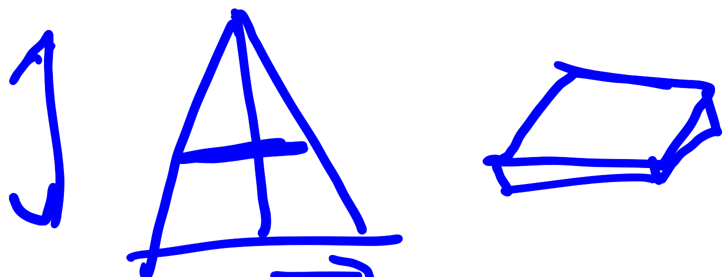
$$= 2\pi\left(1 - 5e^{-4}\right)$$

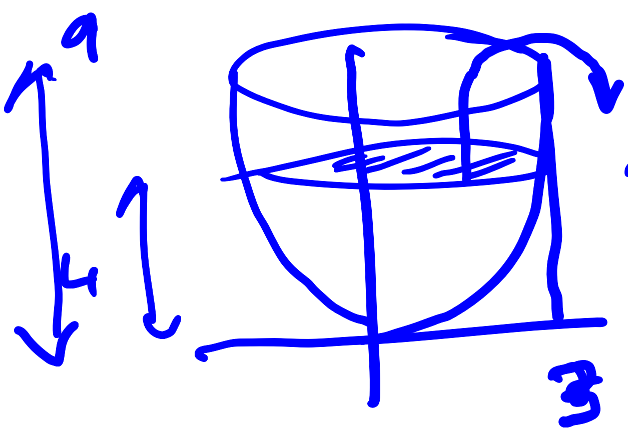
x^2 , x^3







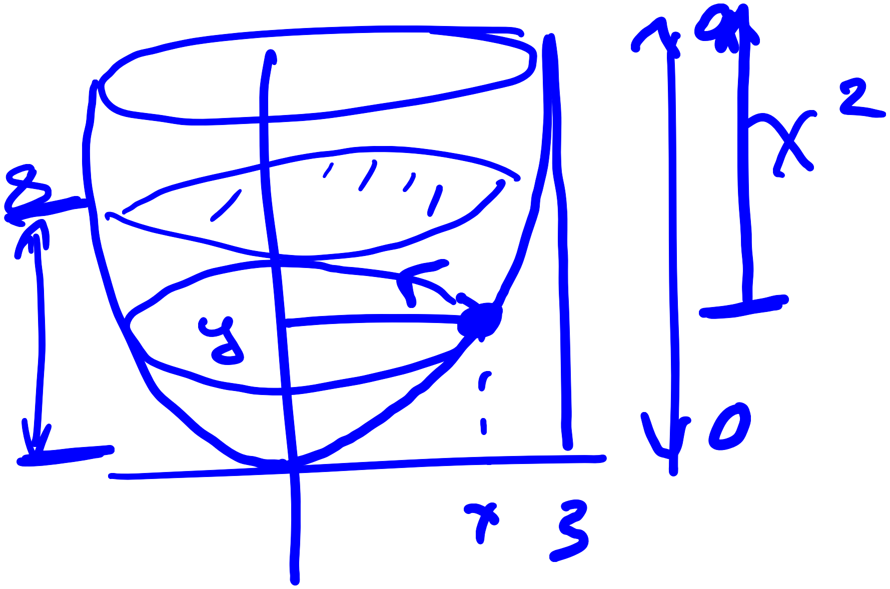




x^2

x^3

x^4



$0 \le y \le 8$

QUIZ 6, Pick

$y = x^2$

$x = \sqrt{y}$

$A = \pi r^2 = \pi (\sqrt{y})^2 = \pi y$

$\text{dist} = 9 - y$

$$W = 62.4 \cdot \underbrace{\int_0^8 \pi y \,(9-y)\,dy}_{\text{compute}}$$

HW 7, Q 16

bucket = 20 kg sand

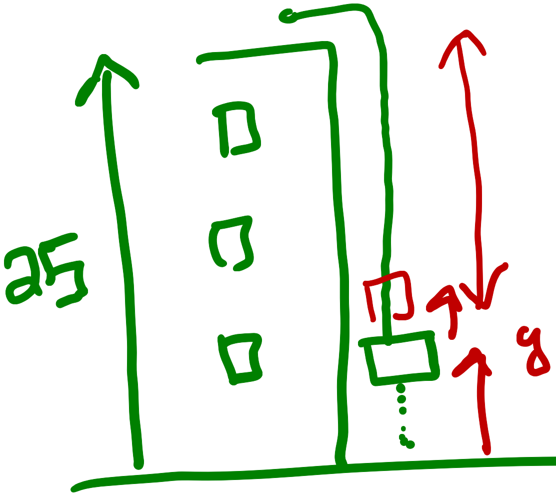
building 25 m

rope = .4 kg/m

1 meter rope to tie knot on bucket

when bucket reaches top, it has 14 kg sand, sand leaks at constant rate.

Find work lifting sand!



end 14 kg mass

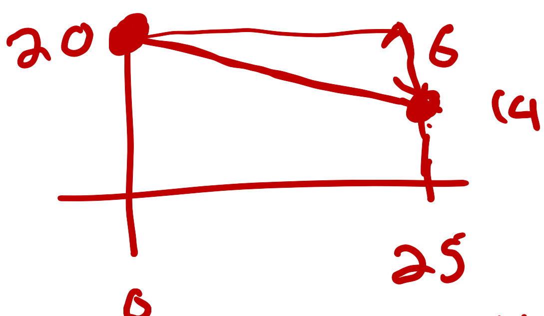
How heavy is rope + bucket at height y?

starts 20 kg mass

bucket $= ? = 20 - \frac{6}{25} y$

knot $= .4$ kg

rope from bucket to roof $= (25 - y)(.4)$



$m = \frac{-6}{25}$

Force $= 9.8 \cdot$ mass in kg.

Total ~~weight~~ mass at height y $= \left(20 - \frac{6}{25} y\right) + .4 + (25 - y) .4$

$$W = 9.8\int_0^{25} F \cdot d$$

$$= 9.8\int_0^{25} \Big[\underbrace{2\cancel{0} - \frac{6}{25}y + .\cancel{4} \to (25-y).4}_{F}\Big]\,dy$$

$$= 9.8\int_0^{25} \Big[(20.4+10) \qquad\qquad \Big]\,dy$$

$$30.4 - \frac{24}{100}y - .4y$$

$$.24$$

$$= 9.8\int_0^{25} 30.4 - .64y \; dy$$

$$= 9.8(25)(30.4) - .64\,\tfrac{1}{2}y^2 \Big|_0^{25}$$

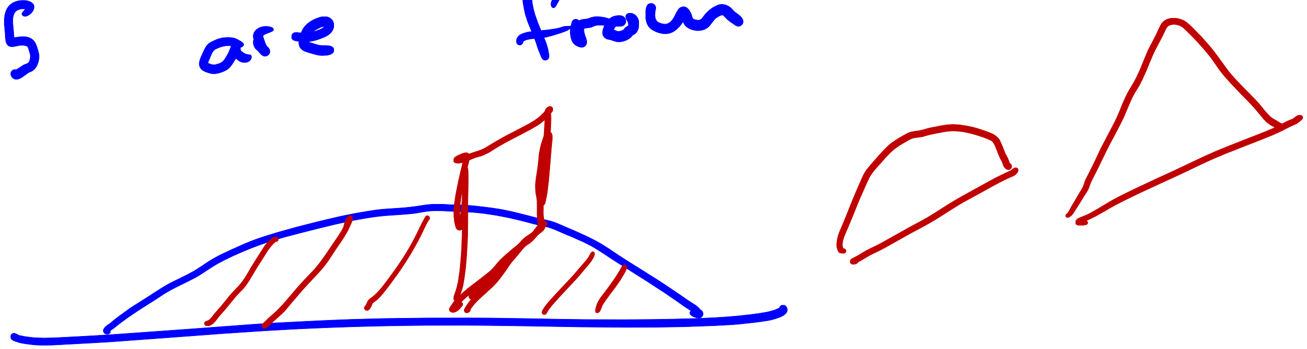
$$= 9.8(25)(30.4) - (.64)\tfrac{1}{2}\cdot 25^2$$

$$= 5488 \text{ joules } ?$$

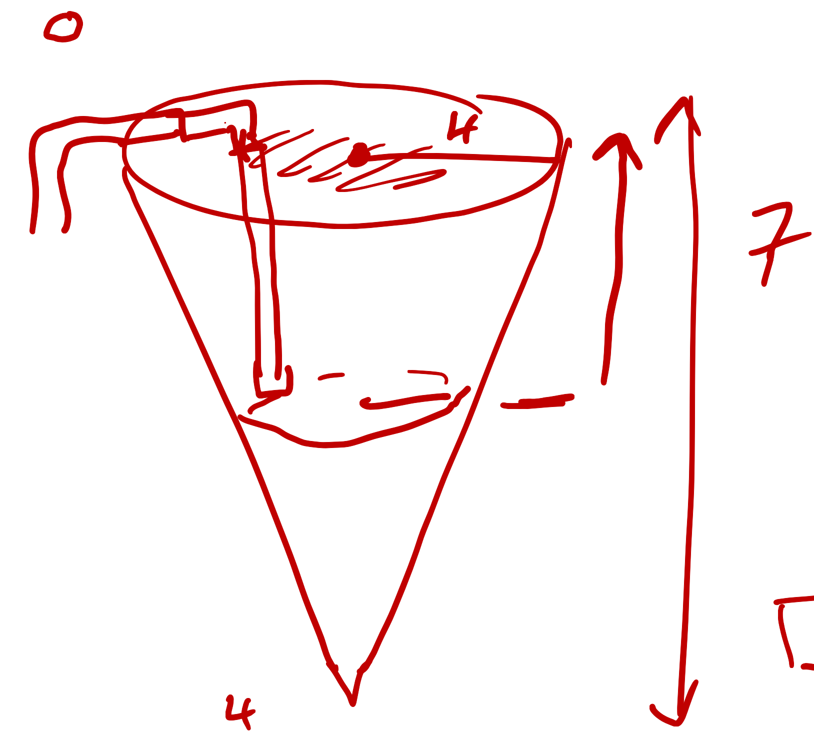
Midterm = Quizzes
+
HW
+
in lecture.

25 ques.

15 are from

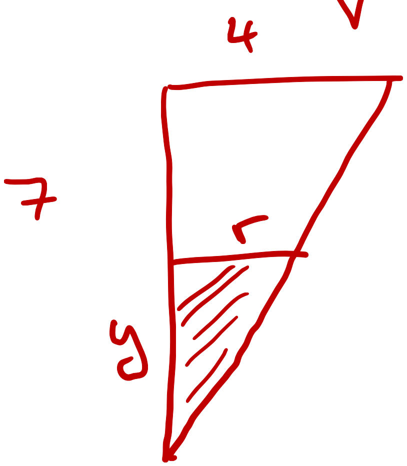


HW 7, Q 20



"by a pipe that is always level at the surface of the water".

$$\square \int_0^7 \underbrace{\pi\left(\frac{4}{7}y\right)^2}_{A}(7-y)\,dy$$



$$\frac{4}{7} = \frac{r}{y}$$

$$r = \frac{4}{7}y$$

$$= \pi \frac{16}{49} \int_0^7 7y^2 - y^3 \, dy$$

$$= \pi \frac{16}{49} \left[\frac{7}{3}y^3 - \frac{1}{4}y^4 \right]_0^7$$

$$= \pi \frac{16}{49} \left[\frac{7^4}{3} - \frac{7^4}{4} \right]$$

$$7 \cdot 7^3$$

$$= \pi \frac{\cancel{16}}{\cancel{49}\cdot\cancel{12}} \cancel{7^2}\cdot 7^2$$

$$= \pi \frac{4}{3}\cdot 7^2$$

$$W = (10000)\cdot \pi \cdot \frac{4}{3}\cdot 49$$

$$= \frac{1960000}{3}\pi \quad \checkmark$$