MAT 126.01, Prof. Bishop, Tuesday, Oct 13, 2020
Section 2.6 Moments and Centers of Mass
Theorem of Pappus

See-Saw example: $m_1 d_1 = m_2 d_2$.

Center of weight of a finite number of masses on line:

$$\overline{x} = \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k} \frac{M}{m}$$

Moments with respect to x-axis and y-axis :

$$M_y = \sum_{k=1}^{n} m_k x_k$$

$$M_x = \sum_{k=1}^{n} m_k y_k$$

$$\overline{x} = \frac{M_y}{m}, \qquad \overline{y} = \frac{M_x}{m}$$

Moment w.r.t. x-axis is on the y-axis and vice versa.

Find center of mass of

$$2 \text{ kg} \quad \text{at } (-1, 3),$$
$$6 \text{ kg} \quad \text{at } (1, 1),$$
$$4 \text{ kg} \quad \text{at } (2, -3),$$

lamina = thin plate represented by region in the plane, constant density.

centroid = center of mass.

Symmetry Principle: if a region is symmetric with respec to a line, then the centroid in on that line

Corollary: if there are two lines of symmetry the centroid is at the intersection point.

Suppose $R = \{(x, y) : a \le x \le b, 0 \le y \le f(x)\}$. Let ρ be the (constant) density of the associated lamina.

$$\text{mass of lamina } = \rho \cdot \text{area}(R) = \rho \cdot \int_a^b f(x)dx$$

$$M_x = \rho \cdot \int_a^b \frac{1}{2}|f(x)|^2 dx$$

$$M_y = \rho \cdot \int_a^b x|f(x)|dx$$

$$\overline{x} = \frac{M_y}{m}, \qquad \overline{y} = \frac{M_x}{m}$$

Find the center of mass of region below $y = \sqrt{x}$ and above $[0, 4]$.

Suppose $R = \{(x,y) : a \le x \le b, g(x) \le y \le f(x)\}$. Let ρ be the (constant) density of the associated lamina.

$$\text{mass of lamina } = \rho \cdot \text{area}(R) = \rho \cdot \int_a^b f(x) - g(x) dx$$

$$M_x = \rho \cdot \int_a^b \frac{1}{2}(|f(x)|^2 - |g(x)|^2) dx$$

$$M_y = \rho \cdot \int_a^b x(f(x) - g(x)) dx$$

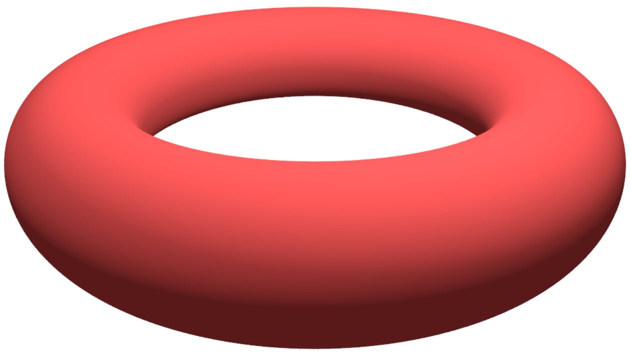
$$\overline{x} = \frac{M_y}{,} \qquad \overline{y} = \frac{M_x}{m}$$

Find the center of mass of the region bounded above by $y = 4 - x^2$ and below by $y = 0$. (Use symmetry).

Find the center of mass of the region bounded above by $y = 1 - x^2$ and below by $y = x - 1$.

Theorem of Pappus: If R is a planar region and L a line not hitting L then the volume formed by rotating R around L is the area of R multiplied by the distance traveled by the center of mass around L.

Let R be the circle of radius 1 centered at $(4, 0)$. What is the volume of the torus formed by rotating this around the y-axis?



Proof of Theorem of Pappus: By method of shells

$$V = 2\pi \int_a^b (f(x) - g(x))dx.$$

$$\text{Area} = m = \int_a^b (f(x) - g(x))dx$$

Distance traveled by center of mass

$$d = 2\pi\overline{x} = 2\pi\frac{M_y}{m} = \frac{2\pi \int_a^b x(f(x) - g(x))dx}{\int_a^b (f(x) - g(x))dx}$$

So $V = dA$.

An equilateral triangle of with side length one is rotated around one of its sides. What is the resulting volume?