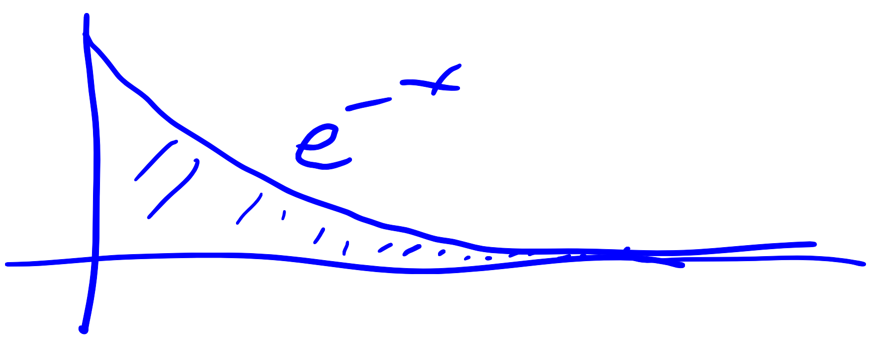
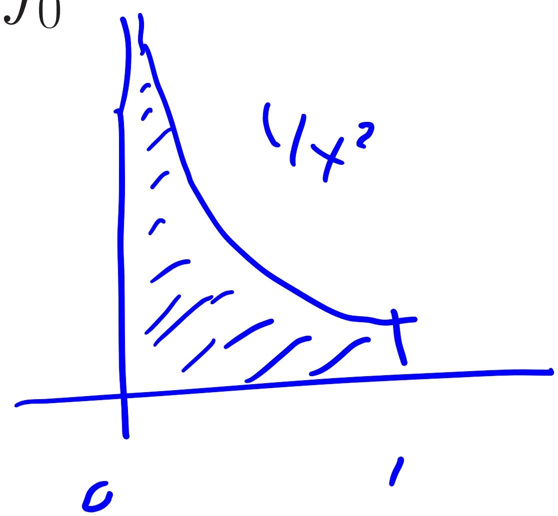
MAT 126.01, Prof. Bishop, Thursday, Nov 5, 2020
Section 3.7: Improper integrals
Quiz 9 review

An integral is **improper** if either the interval is unbounded or the function is unbounded.

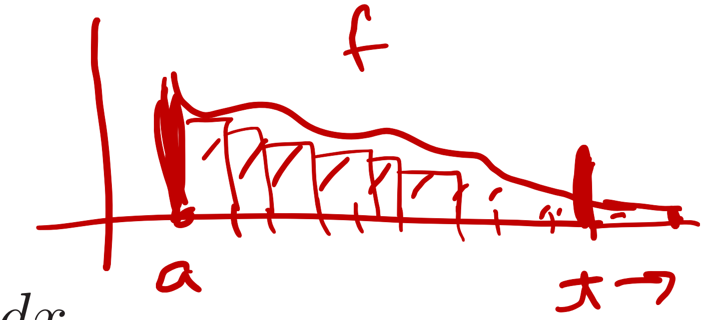
$$\int_0^\infty e^{-x}dx, \qquad \int_{-\infty}^\infty \frac{dx}{1+x^2}$$

$$\int_0^1 \frac{dx}{x^2} \qquad \int_0^{\pi/2} \tan(x)dx$$





Integrals over $[a, \infty)$:



$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

We say integral **converges** if limits exists and is finite. Otherwise the integral **diverges**.

Evaluate $\int_0^\infty e^{-x}dx$.

$$\int_0^\infty e^{-x}dx = \lim_{t\to\infty} \int_0^t e^{-x}dx$$

$$= \lim_{t\to\infty} \left[-e^{-x}\right]_0^t$$

$$= \lim_{t\to\infty} \left[-e^{-t} - (-e^{-0})\right]$$

$$= \lim_{t\to\infty} \left[1 - e^{-t}\right]$$

$$= 1 - \lim_{t\to\infty} e^{-t} = 1 - 0$$

$$\int_0^\infty e^{-x}dx$$

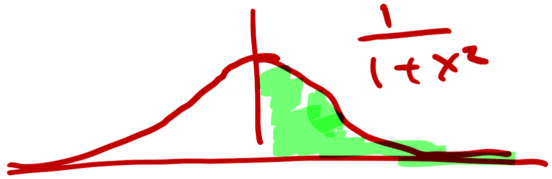
$$-e^{-x}\Big|_0^\infty$$

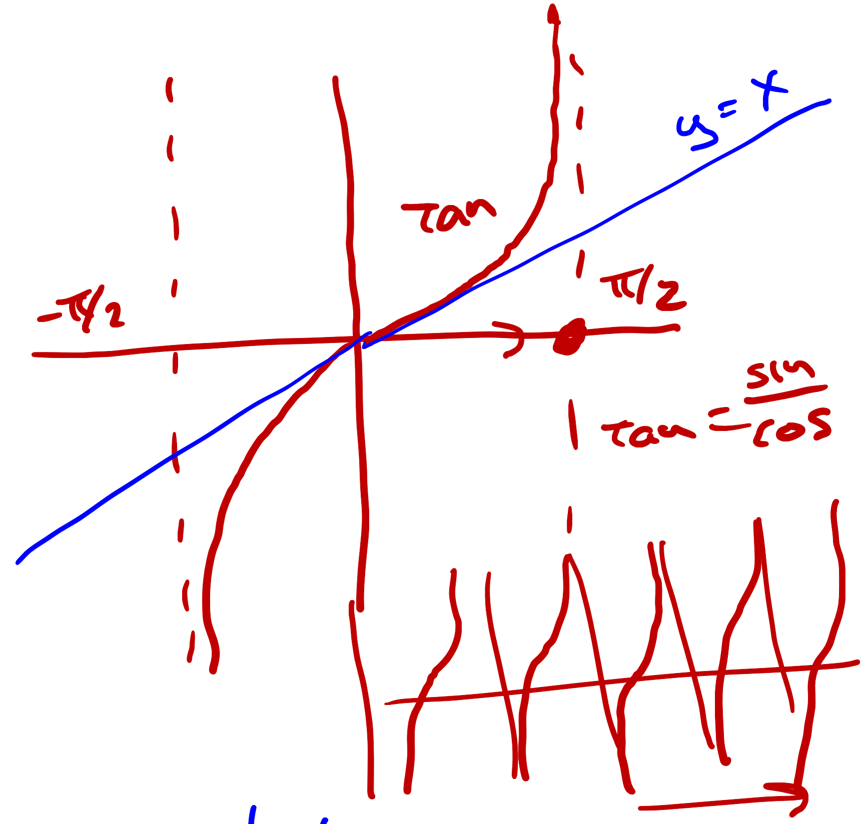
$$-e^{-\infty} - (-e^{-0})$$

$$0 - -1$$

$$= 1$$

Evaluate $\int_0^\infty \frac{dx}{1+x^2}$





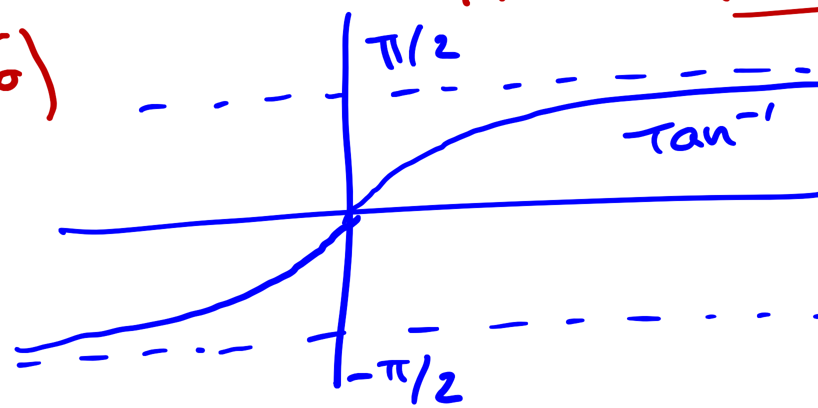
$$= \lim_{t\to\infty} \int_0^t \frac{dx}{1+x^2}$$

$$= \lim_{t\to\infty} \tan^{-1}(x)\Big|_0^t$$

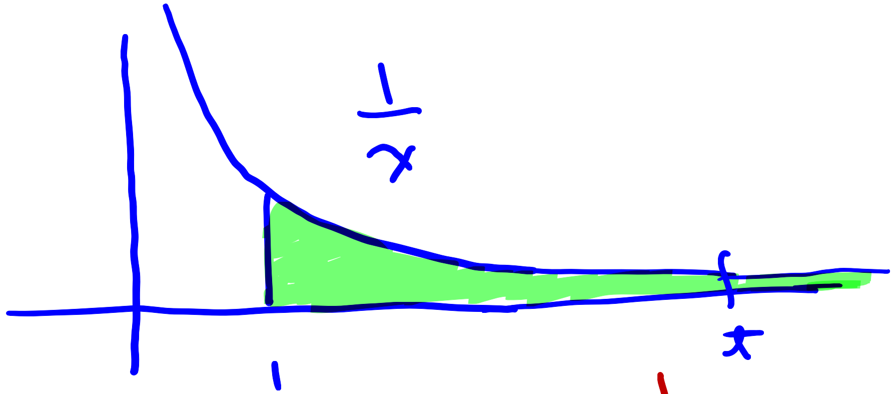
$$= \lim_{t\to\infty} \tan^{-1}(t) - \tan^{-1}(0)$$

$$= \lim_{t\to\infty} \tan^{-1}(t) - 0$$

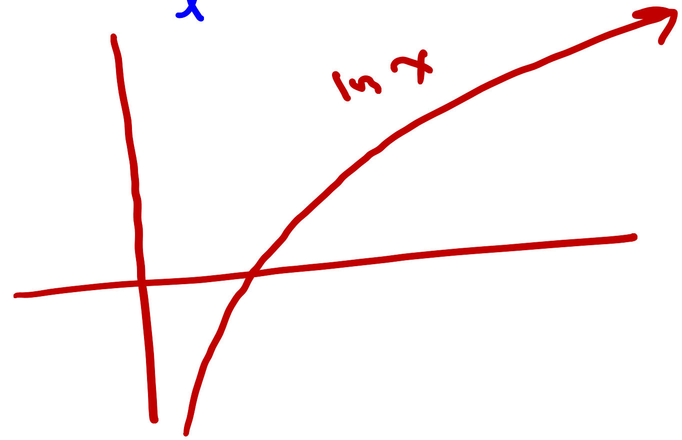
$$\to \pi/2$$



Evaluate $\int_1^\infty \frac{dx}{x}$



$$= \lim_{t \to \infty} \int_1^t \frac{dx}{x}$$



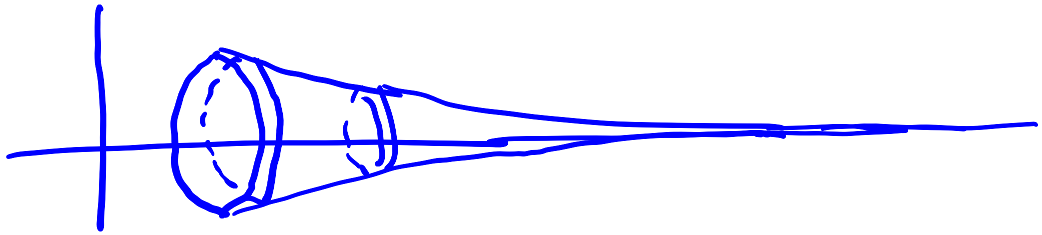
$$= \lim_{t \to \infty} \ln(x)\Big|_1^t$$

$$= \lim_{t \to \infty} \ln(t) - \underbrace{\cancel{\ln(1)}}_{0}$$

$$= \lim_{t \to \infty} \ln(t)$$

$$= \infty$$ Diverges

Find volume of $1/x$ rotated around x-axis for $1 \leq x < \infty$.



$$\int x^n = \frac{x^{n+1}}{n+1}$$

$$\int x^{-2} = \frac{1}{-1} x^{-1}$$

$$\text{vol} = \int \pi f(x)^2 \, dx$$

$$= \int_1^\infty \pi \frac{1}{x^2} \, dx$$

$$= \pi \lim_{t \to \infty} \int_1^t \frac{1}{x^2} \, dx$$

$$= \pi \lim_{t \to \infty} \left(-\frac{1}{x}\right)\Big|_1^t$$

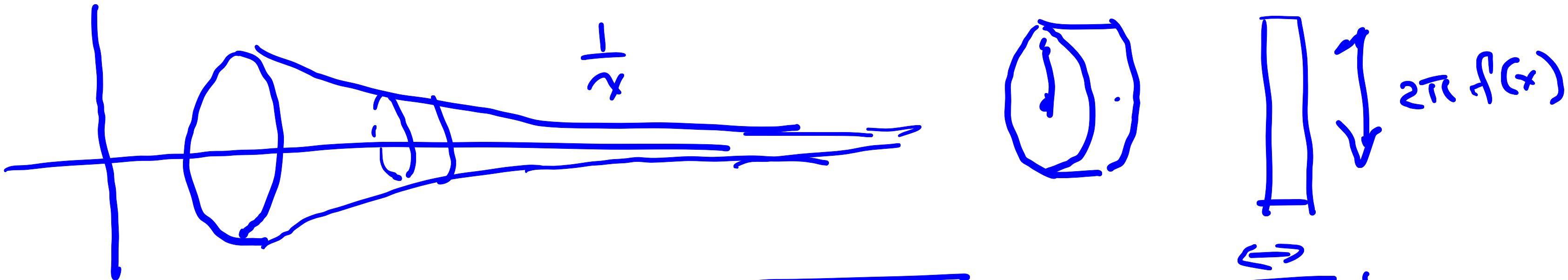
$$= \pi \lim_{t \to \infty} -\frac{1}{t} - \left(-\frac{1}{1}\right)$$

$$= \pi \left(1 - \lim_{t \to \infty} \frac{1}{t}\right)$$

$$= \pi \cdot 1$$

$$= \pi$$

Find surface of $1/x$ rotated around x-axis for $1 \leq x < \infty$.



$$\text{area} = \int 2\pi f(x) \sqrt{1+(f'(x))^2}\, dx \qquad \sqrt{\quad}\, dx$$

$f(x) = \frac{1}{x}$

$f'(x) = -\frac{1}{x^2}$

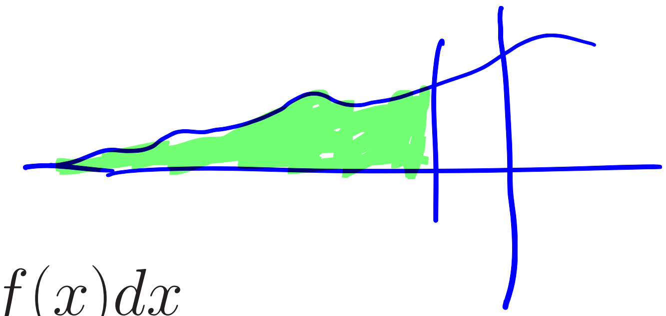
$$= 2\pi \int_1^\infty \frac{1}{x} \underbrace{\sqrt{1+1/x^4}}_{\geq 1}\, dx$$

$$\geq 2\pi \int_1^\infty \frac{1}{x} \cdot 1 \, dx$$

$$= 2\pi \cdot \infty$$

$= \infty$ Diverges, Area = infinite

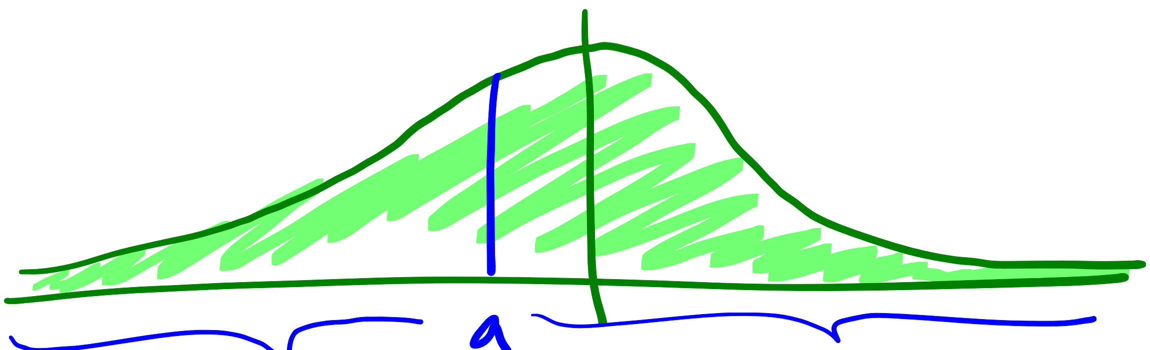
Integrals over $(-\infty, b)$:

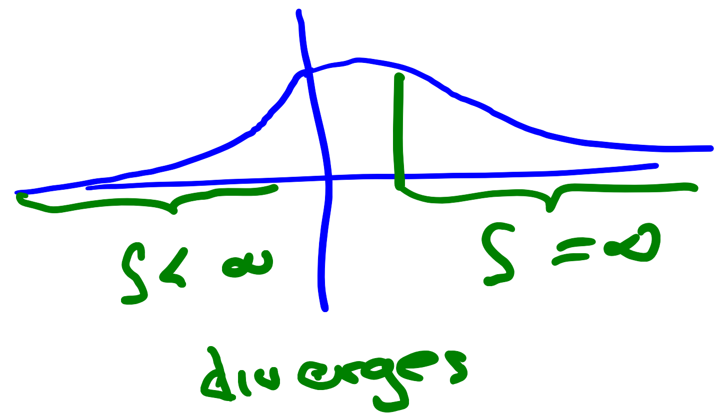


$$\int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty} \int_{-t}^{b} f(x)dx$$

Integrals over $(-\infty, \infty)$:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx$$

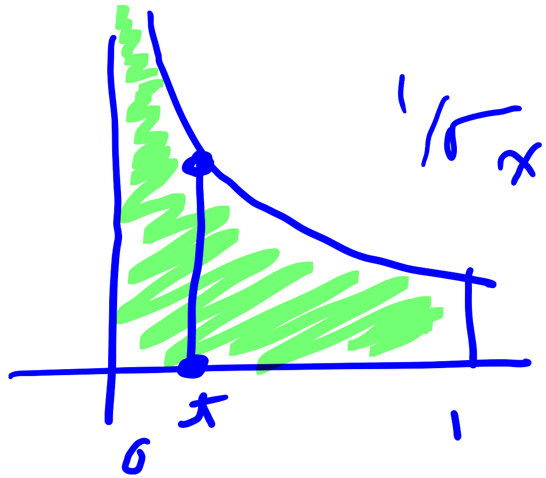




If f is continuous on $(a, b]$ but unbounded as $x \to a$, define

$$\int_a^b f(x)dx = \lim_{t \searrow a} \int_t^b f(x)dx.$$

Evaluate $\int_0^1 \frac{dx}{\sqrt{x}}$



$$1/\sqrt{x} = x^{-1/2}$$

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{t \to 0} \int_t^1 x^{-1/2} dx$$

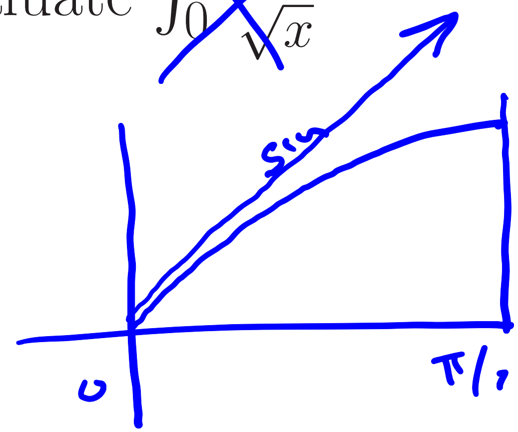
$$= \lim_{t \to 0} 2x^{1/2} \Big|_t^1$$

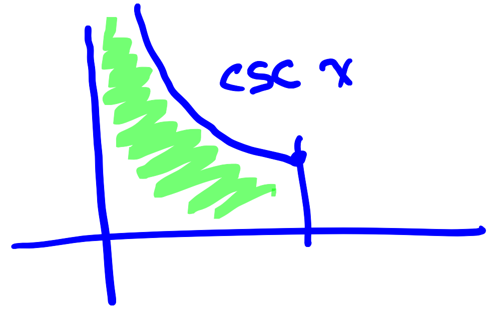
$$= \lim_{t \to 0} \Big[2 - \underbrace{2\sqrt{t}}_{\to 0} \Big]$$

$$= 2$$

Evaluate $\cancel{\int_0^1 \frac{dx}{\sqrt{x}}}$

$$\int_0^{\pi/2} \frac{1}{\sin x}\,dx = \int_0^{\pi/2} \csc x\,dx$$





$\sin x \le x$

$\boxed{\csc x \ge \frac{1}{x}}$

$$\int_0^{\pi/2} \csc x\,dx \ge \int_0^{\pi/2} \frac{1}{x}\,dx$$

$$= \lim_{t\to 0} \int_t^{\pi/2} \frac{1}{x}\,dx$$

$$= \lim_{t\to 0} \left[\ln \frac{\pi}{2} - \ln t\right]$$

$= +\infty$ Divergent

Show $\int_0^1 x^p dx$ diverges if and only if $p < -1$.

if $p = -1$ diverges (last slide)

if $p < -1$

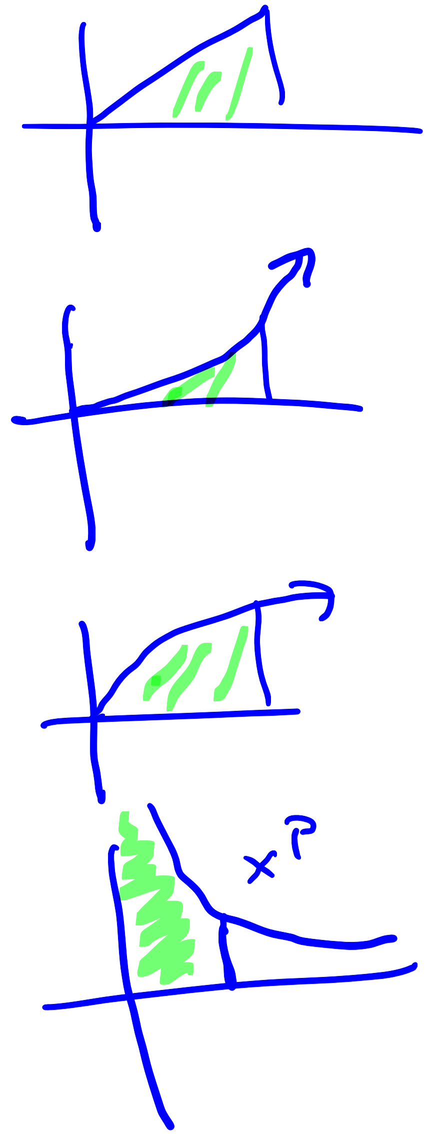
$$x^p > x^{-1} \Rightarrow \text{diverges}$$

if $-1 < p < 0$

$$\lim_{t \to 0} \int_t^1 x^p = \lim_{t \to 0} \left. \frac{x^{p+1}}{p+1} \right|_t^1$$

$$= \frac{1}{p+1} - \lim_{t \to 0} \frac{t^{p+1}}{p+1}$$

$$= \begin{cases} 0 & p > -1 \\ \infty & p < -1 \end{cases}$$



$p = 1$

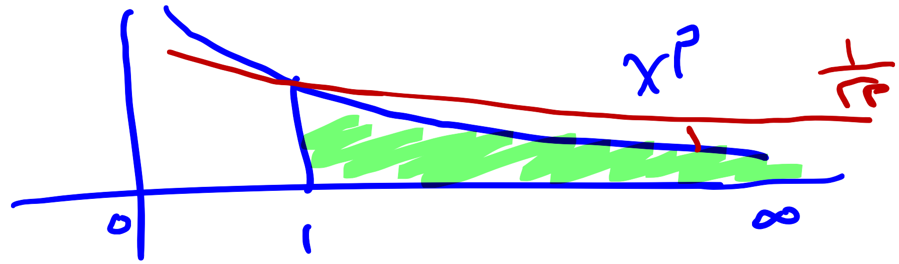
$p > 1$

$0 < p < 1$

$p < 0$

Show $\int_1^\infty x^p dx$ diverges if and only if $p > -1$.

$$\boxed{\begin{array}{ll} p = -1 & \text{diverges} \\ \int_1^t x^{-1} = \ln t \to \infty & \end{array}}$$



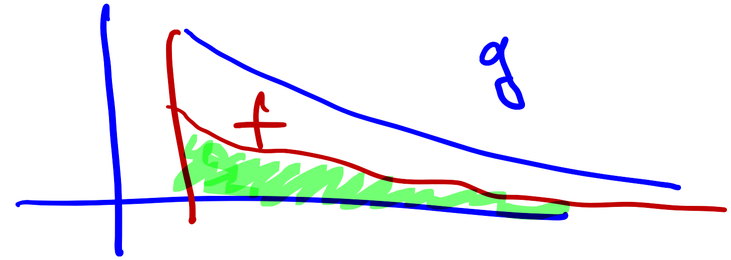
$p > -1 \qquad x^p > x^{-1}$

$\Rightarrow$ diverges.

$p < -1 \qquad \int_1^\infty x^p dx = \lim_{t \to \infty} \left. \frac{x^{p+1}}{p+1} \right|_1^t$

$$= \lim_{t \to \infty} \frac{t^{p+1}}{p+1} - \frac{1}{p+1}$$

$$= \begin{cases} \infty & p > -1 \\ 0 & p < -1 \end{cases}$$



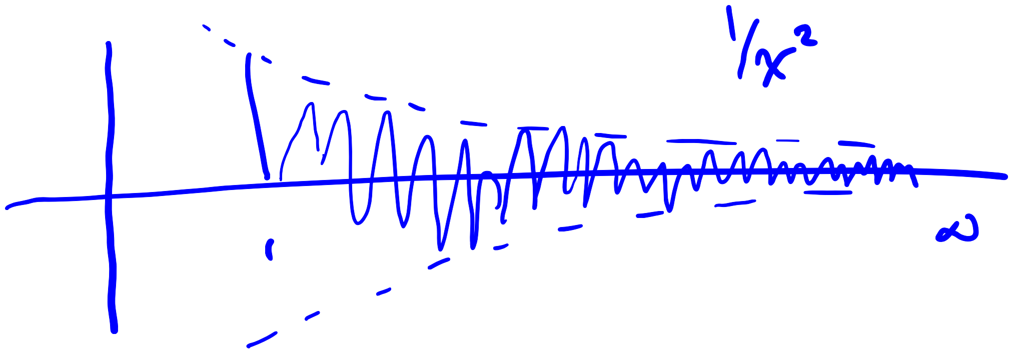
Suppose $0 \leq f(x) \leq g(x)$ on $[a, \infty]$.

If $\int_a^\infty f(x)dx$ diverges, then $\int_a^\infty g(x)dx$ diverges. ✓

If $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ converges. ✓

Even when we can't evaluate exactly, we can often decide if an integral diverges or converges by comparing to a known one.

Does $\int_1^\infty x^{-2}\sin(x^3)dx$ converge or diverge?

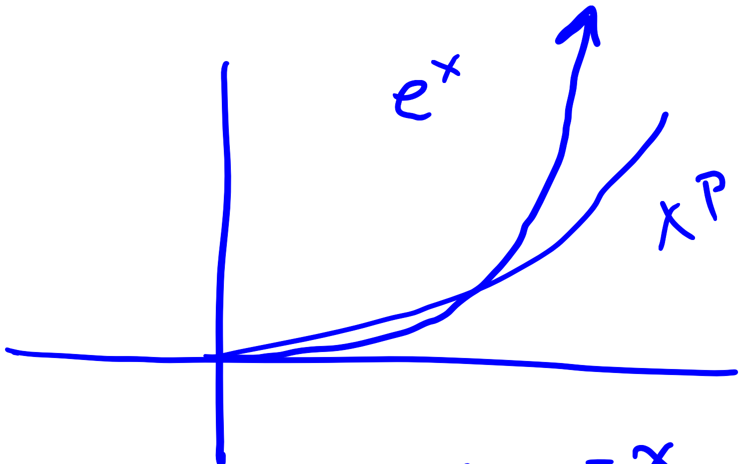


$$\left|\int_1^\infty x^{-2}\sin(x^3)\right| \le \int_1^\infty \frac{1}{x^2}\underbrace{|\sin(x^3)|}_{\le 1}$$

$$\le \int_1^\infty x^{-2}\,dx \qquad -2<-1$$

= Convergent

Does $\int_1^\infty x^{10} e^{-x} dx$ converge or diverge?

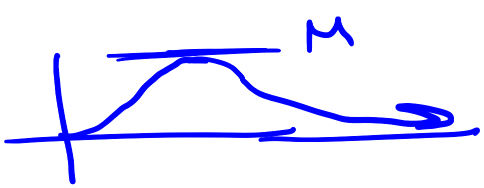


Rule:

$\int_0^\infty p(x) e^{-x} dx$ converges for any polynomial p

$x^{10} e^{-x} \to 0$

$\underbrace{x^{10} \cdot e^{-x/2}}_{\text{bounded}} \cdot e^{-x/2} \le M e^{-x/2}$



$\int_1^\infty x^{10} e^{-x} dx \le M \int_1^\infty e^{-x/2}$

$= M \lim_{t} \int_1^t e^{-x/2}$

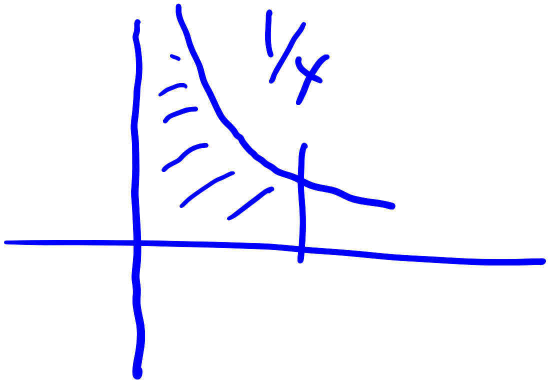
$= M \left(-2e^{-x/2} \Big|_1^t\right)$

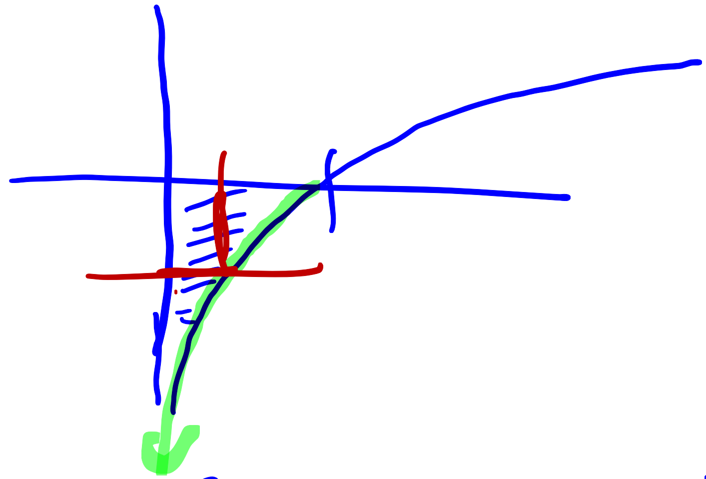
$= M \lim_{t\to\infty} \left(-2e^{-t/2} + 2e^{-1/2}\right)$

$\to 0$

$\boxed{\text{Converges}}$

Does $\int_0^1 \frac{1}{x} \ln x dx$ converge or diverge?





Alternate

$x < \frac{1}{2}$

$\ln x < \ln \frac{1}{2} = -\ln 2$

$\int_0^1 \frac{1}{x} \ln x \le \int_0^1 \frac{1}{x} (-\ln 2) \le -\ln 2 \int_0^1 \frac{1}{x}$

= Diverges

$$\int_0^1 \frac{1}{x} \underbrace{\ln x}_{u} dx = \int u \, du = \frac{1}{2} u^2 = \frac{1}{2} (\ln x)^2$$

$du = \frac{1}{x}$

$$\int_t^1 \frac{1}{x} \ln x \, dx = \lim_{t \to 0} \int_t^1 \frac{1}{x} \ln x \, dx$$

$$= \lim_{t \to 0} \frac{1}{2} (\ln 1)^2 - \frac{1}{2} (\ln t)^2$$

$$= 0 - \frac{1}{2} \lim_{t \to 0} (\ln t)^2$$

= Diverges.

Quiz 9 review: Sections 3.3, 3.4 and 3.7.

Page 1:

- 4 integrals: converge or diverge
- 1 problem: choose correct reference triangle
- 2 integrals: choose correct trig substitution

no evaluation

Page 2:

- choose correct formula from rational function graph
- find partial fraction expansion (non-repeated linear terms)
- long division of polynomials

Write C if the improper integral converges or a D if it diverges (is infinite or undefined).

$\boxed{D}$ $\int_0^1 \frac{dx}{x^2}$ x^p $p=-2<-1$

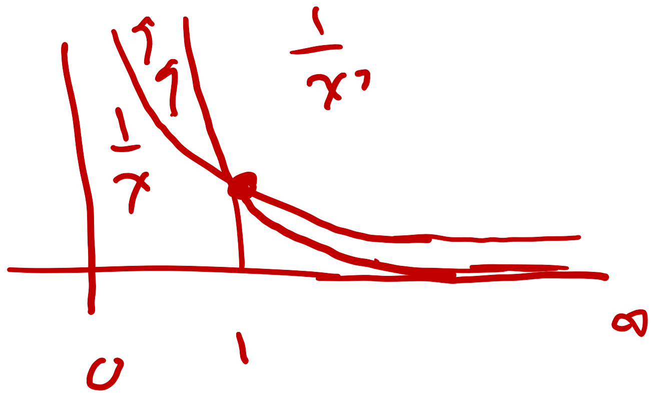
	$(0,1)$	$(1,\infty)$
$p<-1$	Div	Conv
$p>-1$	Conv	Div

$\boxed{C}$ $\int_0^\infty x^8 e^{-8x} dx$

$\boxed{D}$ $\int_0^\infty \frac{x^7+x^4+1}{x^8+3x^4+x^2+5} dx \approx \frac{1}{x}$ $\frac{x^7}{x^8} \frac{(1+x^{-3}+x^{-7})}{(1+3x^{-4}-x^{-6}+5x^{-8})}$

$\frac{1}{x}$ $(\to 1)$

$\boxed{C}$ $\int_{-\infty}^\infty \underbrace{\sin(x)}_{\leq 1} e^{-x} dx$

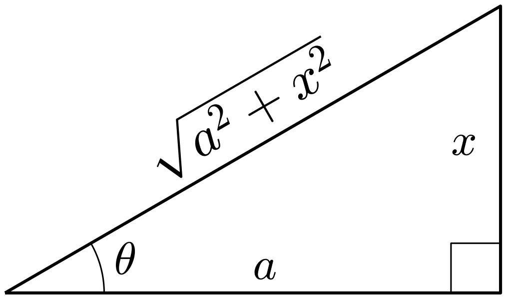


$\int_1^\infty \frac{1}{x^2} < \infty$

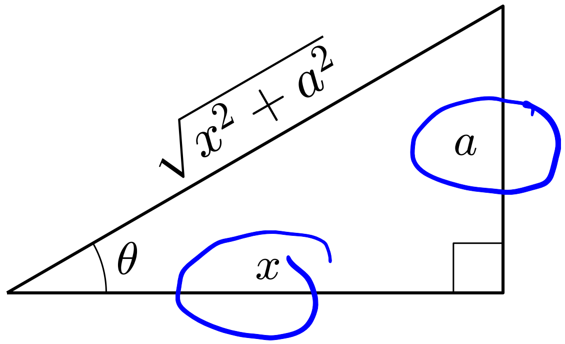
$$\boxed{\frac{x^5+2}{x^8+x^3+2} \approx \frac{1}{x^3} \quad \text{convergent}}$$

$\boxed{B}$ Which reference triangle corresponds to $\cot\theta = \frac{x}{a}$?

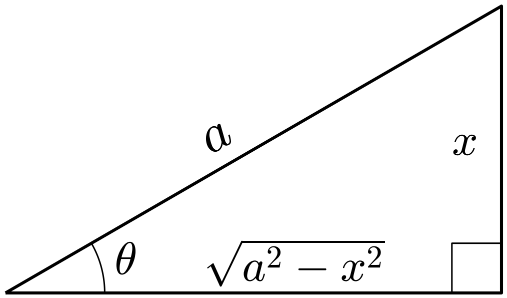
$\frac{\text{adj}}{\text{opp}}$ $\frac{x}{a}$



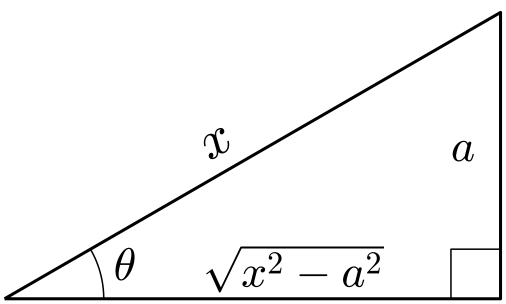
A



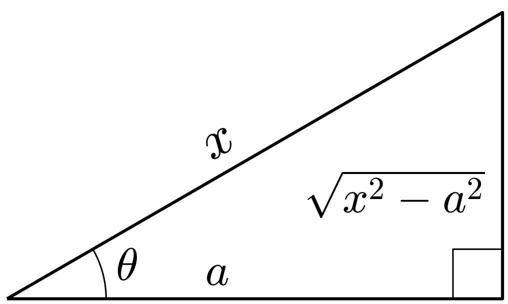
B



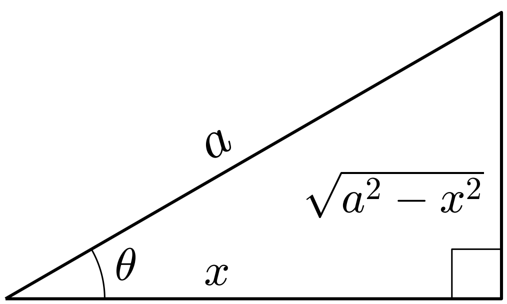
C



D



E



F

For each integral choose the appropriate substitution from the right.

C $\int \frac{dx}{(1-x^2)^{3/2}}$

sin

j $\int \sqrt{x^2+16}dx$

$\sqrt{x^2+a^2}$

$a \tan \theta$

(a) $x = \tan\theta$
(b) $x = \sec\theta$
(c) $x = \sin\theta$
(d) $x = 2\tan\theta$
(e) $x = 2\sec\theta$
(f) $x = 2\sin\theta$
(g) $x = 3\tan\theta$
(h) $x = 3\sec\theta$
(i) $x = 3\sin\theta$
(j) $x = 4\tan\theta$
(k) $x = 4\sec\theta$
(l) $x = 4\sin\theta$

$\boxed{e}$ Which is the partial fraction expansion for the graph below?

(a) $\cancel{\frac{A}{x-4}} + \frac{Bx+C}{(x+4)^2} + \frac{D}{1+(x+1)^2}$

(b) $\cancel{\frac{A}{x+3}} + \frac{Bx+C}{x^2} + \frac{D}{1+(x-3)^2}$

(c) $\cancel{\frac{A}{x-2}} + \frac{Bx+C}{(x+4)^2} + \frac{D}{1+(x+1)^2}$

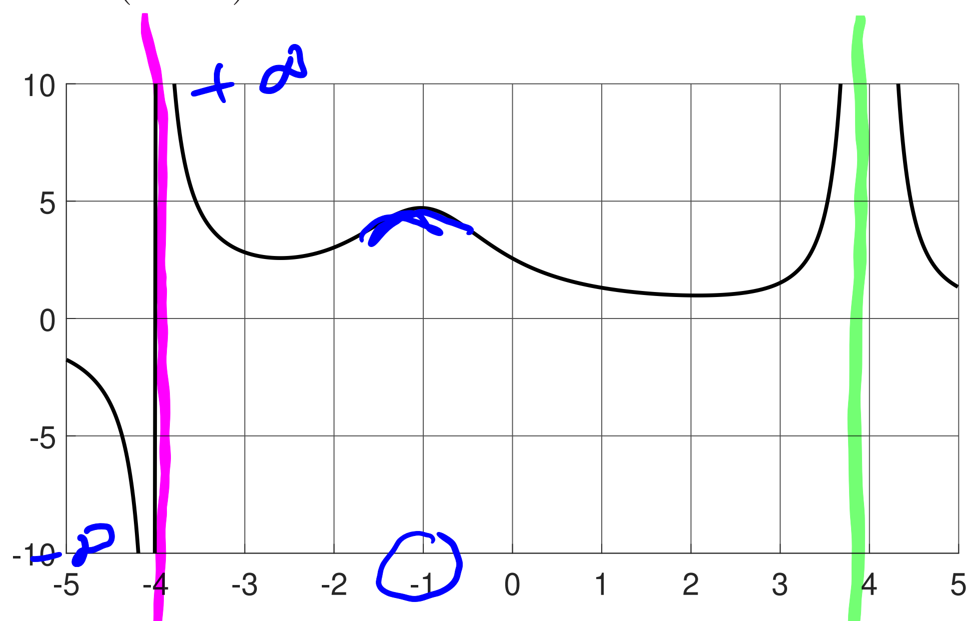
(d) $\cancel{\frac{A}{x+2}} + \frac{Bx+C}{(x-1)^2} + \frac{D}{1+(x-4)^2}$

(e) $\frac{A}{x+4} + \frac{Bx+C}{(x-4)^2} + \frac{D}{1+(x+1)^2}$

(f) $\frac{A}{x+4} + \cancel{\frac{Bx+C}{(x-3)^2}} + \frac{D}{1+x^2}$

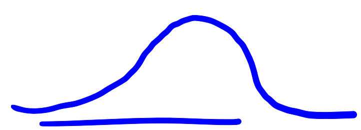
(g) $\cancel{\frac{A}{x+5}} + \frac{Bx+C}{(x+2)^2} + \frac{D}{1+(x-4)^2}$

(h) none of these



$\frac{1}{(x-4)^2}$

$\frac{1}{x+4}$



$\frac{1}{(x+1)^2+1}$

Find A, B where

$$(x^2-1)\quad \underbrace{\frac{9x-1}{x^2-1}} = \underbrace{\left(\frac{A}{x-1} + \frac{B}{x+1}.\right)}(\cancel{x^2-1}) \quad (x-1)(x+1)$$

$x^2-1 = (x-1)(x+1)$

$9x-1 = A(x+1) + B(x-1)$

$= (A+B)x + (A-B)$

$9x-1$

$9 = A+B$

$-1 = A-B$

$8 = 2A$

$$\boxed{\begin{aligned} A &= 4 \\ B &= 5 \end{aligned}}$$

Simplify using long division of polynomials:

$$\frac{4x^3 - x^2 + x}{x^2 - 1}. = \underbrace{4x - 1}_{\text{poly}} + \underbrace{\frac{5x - 1}{x^2 - 1}}_{\text{deg} < \text{deg}}$$

$$R(x) = \frac{P(x)}{Q(x)} \quad \deg(P) < \deg(Q)$$

$$\begin{array}{r} 4x - 1 \\ x^2 - 1 \,\big)\overline{\, 4x^3 - x^2 + x} \\ 4x^3 - 4x \\ \hline 0 \; - x^2 + 5x \\ - x^2 + 1 \\ \hline 0 \quad 5x - 1 \end{array}$$

Nov 5 Office Hours

start $\approx$ 11:20

Quiz 8 : Prob 3 & 4.

(3) Integrate by parts using $dv = 1$

$$\boxed{\int \underbrace{1}_{dv} \cdot \underbrace{\cos(\ln x)}_{u}}\, dx$$

$v = x \qquad du = -\sin(\ln x) \cdot \frac{1}{x}\, dx$

$\int 1 \cos(\ln x)\,dx = uv - \int v\,du$

$= \boxed{\cos(\ln x)\cdot x + \int \cancel{x} \sin(\ln x)\frac{1}{\cancel{x}}dx}$

ans for #3

(4) Apply I. by P. again and solve for integral

$= \cos(\ln x)\cdot x + \int \underbrace{1}_{dv} \underbrace{\sin(\ln x)}_{u}\,dx$

$v = x \quad du = \cos(\ln x)\frac{dx}{x}$

$= \cos(\ln x)\cdot x + x\sin(\ln x) - \int \cancel{x}\cos(\ln x)\frac{1}{\cancel{x}}dx$

$= x\cos(\ln x) + x\sin(\ln x) - \underbrace{\int \cos(\ln x)\,dx}$

$\boxed{\cancel{2}\int \cos(\ln x)\,dx = \frac{x\cos(\ln x) + x\sin(\ln x)}{2}}$

$$= \frac{1}{2}\left(\cos(\ln x) + \cancel{x}\left(\cancel{-\sin(\ln x)}\right)\frac{1}{\cancel{x}}\right.$$

$$\left. + \cancel{\sin(\ln x)} + \cancel{x}\cos(\ln x)\frac{1}{\cancel{x}}\right)$$

$$= \cos(\ln x)$$

HW 10 # 24

Find length of curve $\ln(\sin(x))$

$\frac{\pi}{3} \le x \le \pi/2$

$$\text{arclength} = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

$$= \int_{\pi/3}^{\pi/2} \sqrt{1+\cot^2 x}\;dx$$

$$= \int_{\pi/3}^{\pi/2} \sqrt{\csc^2 x}\;dx$$

$$= \int_{\pi/3}^{\pi/2} \csc x\;dx$$

$f = \ln(\sin(x))$

$f'(x) = \frac{1}{\sin(x)} \cdot \cos(x)$

$= \cot x$

$$\frac{\sin^2}{\sin^2} + \frac{\cos^2}{\sin^2} = \frac{1}{\sin^2}$$

$$1 + \cot^2 = \csc^2$$

$\int \csc x$

$= -\ln|\csc - \cot x|$

$\frac{\pi}{2} = 90^\circ$ $\quad \sin = 1$

$\csc = 1$

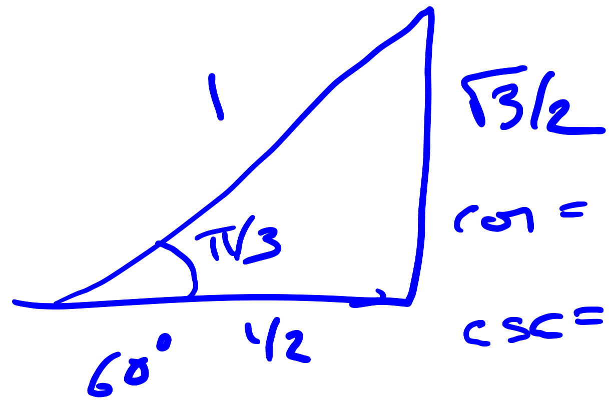
$\cot = 1$

$= \ln|\csc x - \cot x| \Big|_{\pi/3}^{\pi/2}$

$= \ln|1 - 0|$

$- \ln\left|\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right|$

$= \ln(1) - \ln\sqrt{3}$



$\cot = \frac{1}{\sqrt{3}}$ ✓

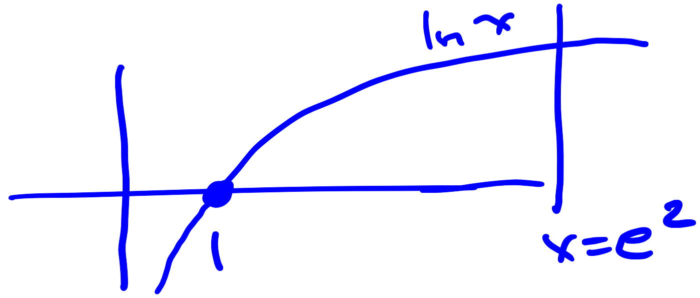
$\csc = \frac{2}{\sqrt{3}}$ ✓

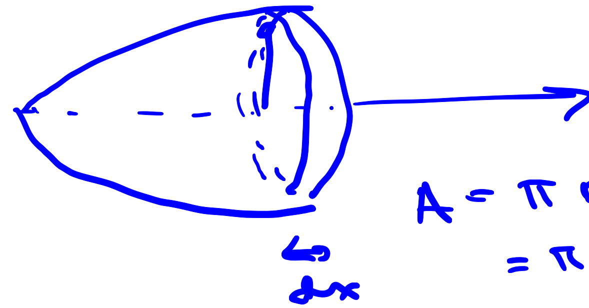
Lumen $= \frac{1}{2}\ln 3 = \ln\sqrt{3}$

$= \ln\sqrt{3} = \boxed{\frac{1}{2}\ln 3}$ ✓

HW 10 #10

$y = \ln x$





$A = \pi r^2$
$= \pi f(x)^2$

$$\text{vol} = \int_a^b \pi f(x)^2 \, dx$$
$$= \int_1^{e^2} \pi \ln^2 x \, dx$$

$$\int \ln^2 x \, dx$$

$$\int \underbrace{1}_{dv} \cdot \underbrace{\ln^2 x}_{u} \, dx = x \ln^2 x - \int \cancel{x} \, 2 \ln x \, \cancel{\tfrac{1}{x}} \, dx$$

$v = x \qquad du = 2 \ln x \cdot \frac{1}{x}$

$$= x \ln^2 x - 2 \int \ln x \, dx$$
$$= x \ln^2 x - 2(x \ln x - x)$$
$$= x \ln^2 x - 2x \ln x + 2x$$

$$\text{vol} = \pi \int_1^{e^2} (\quad) = \pi \cdot \Big[x \ln^2 x - 2x \ln x + 2x \Big] \Big|_1^{e^2}$$
$$= \pi \Big[e^2 (\ln e^2)^2 - 2e^2 \ln e^2 + 2e^2 \Big]$$
$$- \Big[1 \ln^2 1 - 2 \cdot \ln 1 + 2 \Big]$$

$= \pi \left[4\cancel{e^2} - 4\cancel{e^2} + 2e^2 \right] - \left[0\,0 + 2 \right]$

$= \pi \left(2e^2 - 2 \right)$

$= 2\pi \left(e^2 - 1 \right)$ ✓

Lumen

$= 2\pi \left(e^2 - 1 \right)$

HW 10 # 22

$v(t) = \sin(\omega t)\cos^6(\omega t)$

Find position if $x = f(t)$, $f(0) = 0$

$p = \int v$

$= \int_0^t \underbrace{\sin(\omega x)}\cos^6(\omega x)\,dx$

$u = \cos(\omega x)$

$du = -\sin(\omega x)\cdot\omega$

$= -\frac{1}{\omega} \int u^6\,du$

$$= -\frac{1}{\omega}\,\frac{1}{7}\,u^7$$

$$= -\frac{1}{\omega}\,\frac{1}{7}\,\cos^7 \omega x \Big|_0^{\pi}$$

$$= -\frac{1}{7\omega}\left(\cos^7 \omega\pi - 1\right)$$

$$= \frac{1 - \cos^7 \omega\pi}{7\omega}$$

$$\int \sin^k \cos^l$$

$$\int \sin^5 \cos^3$$

$$\int \sin^5 (1-\sin^2) \cos$$

$$\int \sin^5 - \sin^7 \ \cos$$

$$u^5 - u^7 \ du$$

$$= \frac{1}{6}\sin^6 - \frac{1}{7}\sin^7$$

$$\tan^6 \sec^2$$

$$(\tan)' = \sec^2$$

$$\int u^6 \, du$$

$$u = \tan$$