MAT 126.01, Prof. Bishop, Tuesday, Nov 3, 2020
Recorded lecture
Finish Section 3.3: Trigonometric substitution
Section 3.4: Partial fractions

1. No live lecture on Nov 3.
   office hours canceled.

2. You may use 1 page notes on Quiz 8.
   Sample sheet posted on webpage.

Section 3.3: Integration involving $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$.

Some such integrals we can already do:

$$\underbrace{\int \frac{dx}{\sqrt{a^2-x^2}}}, \qquad \underbrace{\int \frac{x\,dx}{\sqrt{a^2-x^2}}}_{\text{sub}}, \qquad \underbrace{\int x\sqrt{a^2-x^2}\,dx}_{\text{sub}}$$

In general, for integrals involving:

$\sqrt{a^2 - x^2}$ use $x = a\sin\theta$, $dx = a\cos\theta d\theta$. $\sqrt{a^2 - a^2\sin^2} = a\cos\theta$

$\sqrt{a^2 + x^2}$ use $x = a\tan\theta$, $dx = a\sec^2\theta d\theta$.

$\sqrt{x^2 - a^2}$ use $x = a\sec\theta$, $dx = a\sec\theta\tan\theta d\theta$.

Last week (Oct 29) we did the first case.

Integrals involving $\sqrt{a^2+x^2}$, use substitution $x = a\tan\theta$.

Evaluate $\int \frac{dx}{\sqrt{1+x^2}}$

$a = 1$

$x = \tan\theta \qquad dx = \sec^2\theta \, d\theta$

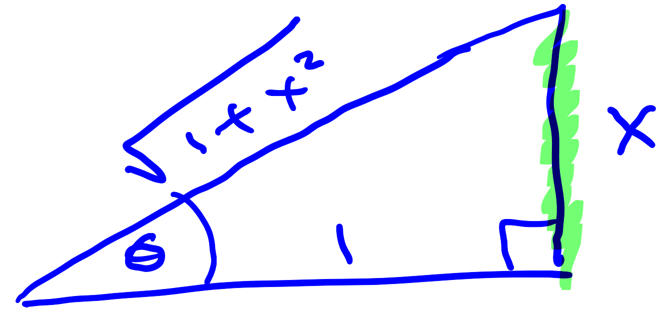
$= \int \frac{\sec^2\theta \, d\theta}{\sqrt{1+\tan^2\theta}}$

$= \int \frac{\sec^{\cancel{2}}\theta \, d\theta}{\cancel{\sec\theta}}$

$= \int \sec\theta \, d\theta$

$= \ln|\sec\theta + \tan\theta| + C$

$\boxed{= \ln\left|\sqrt{1+x^2} + x\right| + C}$



$x = \tan\theta$

$\tan\theta = \frac{\text{opp}}{\text{adj}}$

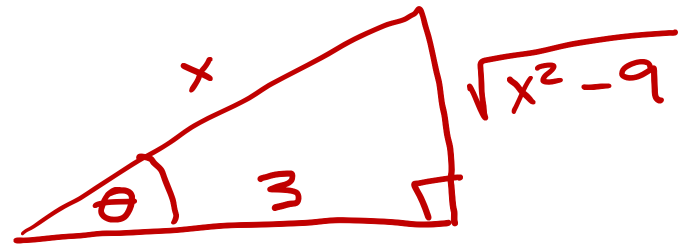
$\tan\theta = \frac{x}{1} = x$

$\sec\theta = \frac{1}{\cos} = \frac{\text{hyp}}{\text{adj}}$

$= \frac{\sqrt{1+x^2}}{1}$

Integrals involving $\sqrt{x^2 - a^2}$, use substitution $x = a\sec\theta$.

Evaluate $\int \sqrt{x^2 - 9}dx$



$a = 3, \quad x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta\, d\theta$

$= \int \sqrt{9\sec^2\theta - 9}\quad 3\sec\theta\tan\theta\, d\theta$

$= 9\int \sqrt{\tan^2\theta} \cdot \sec\theta\tan\theta\, d\theta$

$= 9\int \tan^2\theta \sec\theta\, d\theta$

$= 9\int (\sec^2\theta - 1)\sec\theta\, d\theta$

$= 9\int \sec^3\theta - \sec\theta\, d\theta$

$= \frac{9}{2}\ln\left[\sec\theta + \tan\theta\right] + \frac{9}{2}\sec\theta\tan\theta - 9\ln|\sec\theta - \tan\theta| + C$

$= \frac{9}{2}\ln\left[\frac{x}{3} + \frac{1}{3}\sqrt{x^2-9}\right] + \frac{9}{2}\,\frac{x}{3}\,\frac{1}{3}\sqrt{x^2-9} - 9\ln\left|\frac{x}{3} - \frac{1}{3}\sqrt{x^2-9}\right| + C$

$\sec\theta = \frac{x}{3}$

$\frac{hyp}{adj} = \frac{x}{3}$

$\sec\theta = \frac{x}{3}$

$\tan\theta = \frac{opp}{adj}$

$= \frac{\sqrt{x^2-9}}{3}$

Section 3.4 Partial fractions:

Polynomial functions:

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

. x^2+1, $3x^3+2x^2+x+5$, $\pi x^2+\sqrt{2}\,x$, $\cancel{x+\sqrt{x}}$ not poly

A rational function is a ratio of polynomials:

$\frac{x^2}{1+x}$, $\frac{x^3+x^2+1}{x^4+x^3+5}$

$$R(x) = \frac{P(x)}{Q(x)}.$$

$\frac{4}{3} = 1\,1/3$

Using long division we can always write a rational function as a polynomial plus a rational function where the numerator has smaller degree than the denominator.

degree = highest power of x

$\deg(x^2+x+1) = 2$ $\deg(3x^5+x^4+x^2) = 5$

Long division of numbers

$$\frac{300}{14}$$

$$\begin{array}{r@{\,}l}
 & 21 \\
\cline{2-2}
14 & \big)\,300. \\
 & \,28 \\
\cline{2-2}
 & \,20 \\
 & \,14 \\
\cline{2-2}
 & \,6
\end{array}$$

$$\frac{300}{14} = \boxed{21 + \frac{6}{14}}$$

$$= 21\frac{6}{14}$$

Long division of polynomials

Simplify $\frac{x^2+3x+5}{x+1}$.

$$
\begin{array}{r}
x+2 \\
x+1 \,\overline{\big)\, x^2+3x+5} \\
x^2+x \\
\hline
0 \quad 2x+5 \\
2x+2 \\
\hline
0 \quad 3
\end{array}
$$

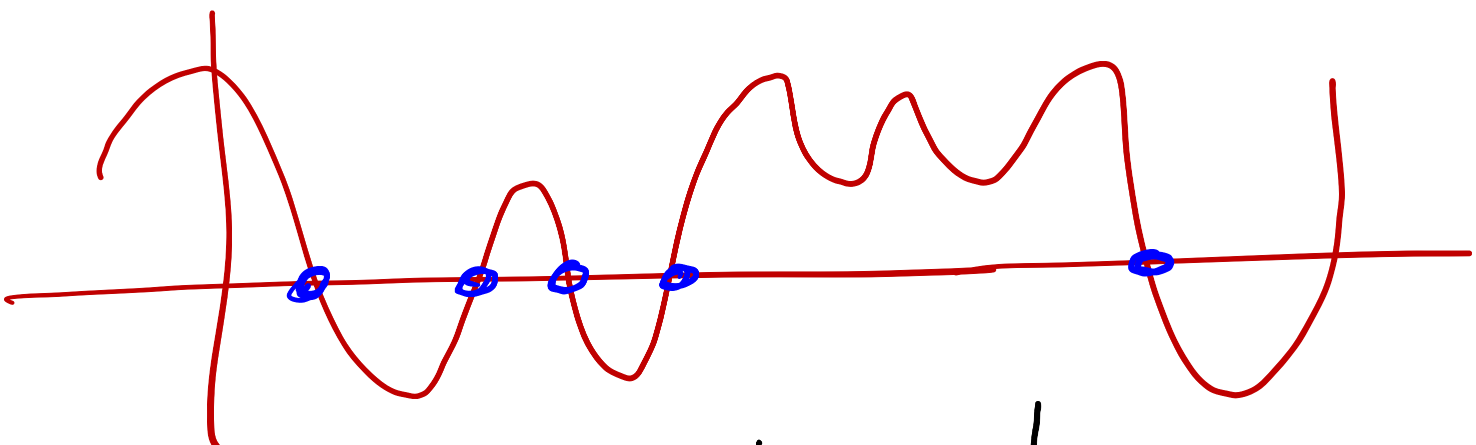
$$\frac{x^2+3x+5}{x+1} = \underbrace{(x+2)} + \frac{3}{x+1}$$

$$
\begin{array}{r}
x^2+2x-2 \\
x^2+x \,\overline{\big)\, x^4+3x^3+0x^2+0x+3} \\
x^4+x^3 \\
\hline
0 \quad 2x^3+0 \\
2x^3+2x^2 \\
\hline
0 \quad -2x^2+0 \\
-2x^2-2x \\
\hline
0 \quad 2x+3
\end{array}
$$

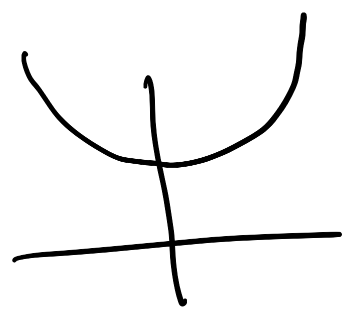
$$\frac{x^4+3x^3+3}{x^2+x} = \underbrace{x^2+2x-2} + \underbrace{\frac{2x+3}{x^2+2}}$$

Fundamental theorem of algebra: Any polynomial Q with real coefficients can be factored into linear and quadratic factors:

$$Q(x) = (a_1x + b_1)\cdots(a_kx + b_k)(c_1x^2 + d_1x + e_1)\cdots(c_jx^2 + d_jx + e_j).$$



x^2+1

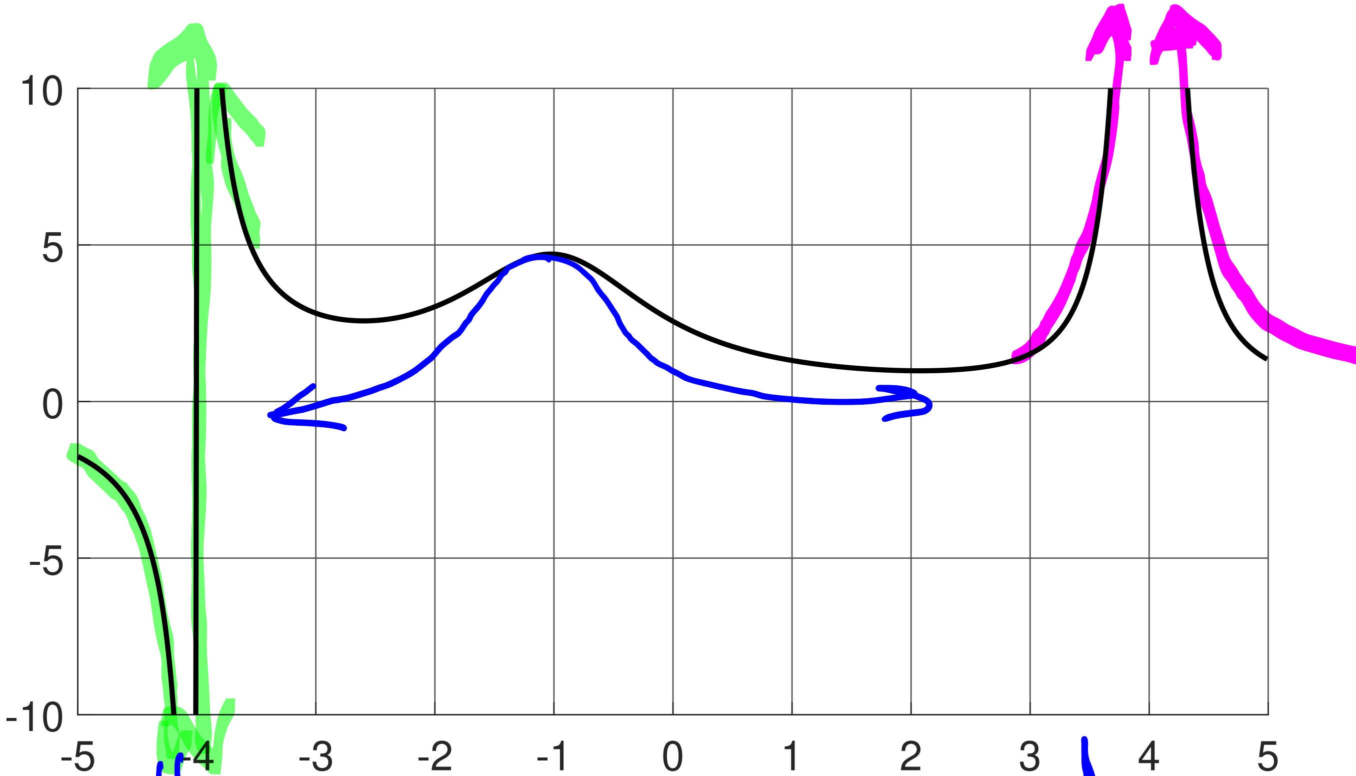


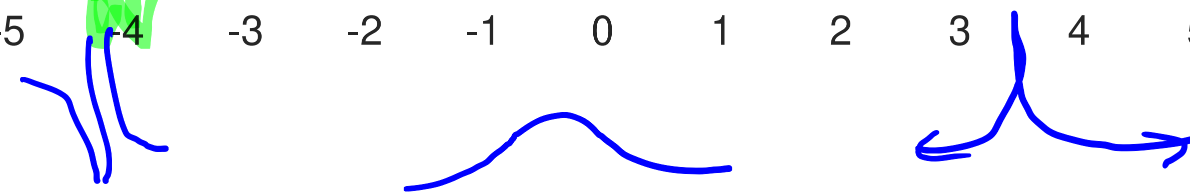
$$\int \frac{P(x)}{Q(x)}$$

Partial fraction expansion (no repeated factors):

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x + b_1)} + \cdots + \frac{A_k}{(a_kx + b_{)}}$$
$$+\frac{C_1x + D_1}{c_1x^2 + d_1x + e_1} + \cdots + \frac{C_jx + D_j}{c_jx^2 + d_jx + e_j}.$$

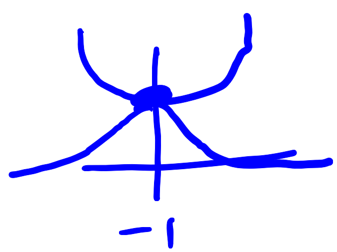
Quiz 9





$$\underbrace{\frac{1}{x+4}}_{x=-4} + \underbrace{\frac{1}{(x+1)^2+1}} + \underbrace{\frac{1}{(x-4)^2}}_{x=4}$$

1/0



To integrate a rational function $R = P/Q$ we usually want to factor Q and use partial fractions.

By long division we can assume $\deg(P) < \deg(Q)$.

Easiest case is when Q has only linear factors and no repeats. If

$$Q(x) = (a_1x + b_1) \cdots (a_kx + b_k),$$

then we can write

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \cdots + \frac{A_k}{a_kx + b_k}$$

Evaluate $\int \frac{3x+2}{x^3-x^2-2x}dx$.

Step 1: factor the denominator.

$$x^3 - x^2 - 2x = x\left(x^2 - x - 2\right)$$
$$= x(x-2)(x+1)$$

Step 2: solve for partial fraction coefficients.

First method: equate coefficients of powers of x:

$$\cancel{(x)}\cancel{(x-2)}\cancel{(x+1)} \quad \frac{3x+2}{\cancel{x}\cancel{(x-2)}\cancel{(x+1)}} = \left[\frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}\right] (x)(x-2)(x+1)$$

$$3x+2 = A(x-2)(x+1) + Bx(x+1) + Cx(x-2)$$

$$3x+2 = A(x^2 - x - 2) + B(x^2 + x) + C(x^2 - 2x)$$

$$3x+2 = x^2(A+B+C) + x(-A+B-2C) + 1(-2A)$$

$$0 = A+B+C$$

$$3 = -A+B-2C$$

$$2 = -2A$$

$$\boxed{A=-1}$$

$$1 = B+C$$

$$2 = B-2C$$

$$1 = -3C$$

$$\boxed{C = -1/3}$$

$$1 = B - 1/3$$

$$\boxed{B = 4/3}$$

2nd method: strategic substitution. Set x = roots of factors:

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}$$

$$3x+2 = A(x-2)(x+1) + Bx(x+1) + Cx(x-2)$$

$x=0$ $\quad 3\cdot 0+2$

$2 = A(-2)(1) + \cancel{B\cdot 0(0+1)} + \cancel{C\cdot 0(0-2)}$

$2 = -2A$

$\boxed{-1 = A}$

$x=2$ $\quad 8 = \cancel{A\cdot 0} + B\cdot 2\cdot 3 + \cancel{C\cdot 2\cdot 0}$

$8 = 6B$

$\boxed{B = 4/3}$

$x=-1$ $\quad -1 = A\cdot 0 + B\cdot 0 + C(-1)(-3)$

$\boxed{C = -1/3}$

Step 3: integrate each term.

$$\int \frac{3x+2}{x^3-x^2-2x}\,dx = \int \frac{-1}{x} + \frac{4}{3}\frac{1}{x-2} - \frac{1}{3}\frac{1}{x+1}\,dx$$

$$= -\ln|x| + \frac{4}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C$$

Partial fractions with repeated linear factors:

$$\frac{P(x)}{(ax+b)^n} = \underline{\frac{A_1}{ax+b}} + \underline{\frac{A_2}{(ax+b)^2}} \cdots + \underline{\frac{A_n}{(ax+b)^n}}.$$

Evaluate $\int \frac{x-2}{(2x-1)^2(x-1)}dx$.

$$= \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{x-1}$$

$$x-2 = A(2x-1)(x-1) + B(x-1) + (2x-1)^2 C$$

$x = \frac{1}{2}$ $\qquad -\frac{3}{2} = A \cdot 0 + B\left(-\frac{1}{2}\right) + 0 \cdot C$

$$\boxed{B = 3}$$

$x = 1$ $\qquad -1 = A \cdot 0 + B \cdot 0 + 1 \cdot C$

$$\boxed{-1 = C}$$

$x = 0$

$$-2 = A(-1)^2 + B(-1) + (2\cdot 0 - 1)^2 C$$

$$= A - B + C$$

$$-2 = A - 3 - 1$$

$$\boxed{2 = A}$$

$$\int \frac{x-2}{(2x-1)^2(x-1)}\,dx = \int \frac{2}{2x-1} + \int \frac{3}{(2x-1)^2} + \int \frac{-1}{x-1}$$

$$= \ln|2x-1| + \frac{-3}{2(2x-1)} - \ln|x-1|$$

$$= \ln|2x-1| - \frac{3}{2}\frac{1}{(2x-1)} - \ln|x-1| + C$$

Partial fractions with quadratic term: If Q contains a non-repeated irreducible quadratic factor $ax^2 + bx + c$ (no real roots), then the partial fraction expansion contains a factor of the form

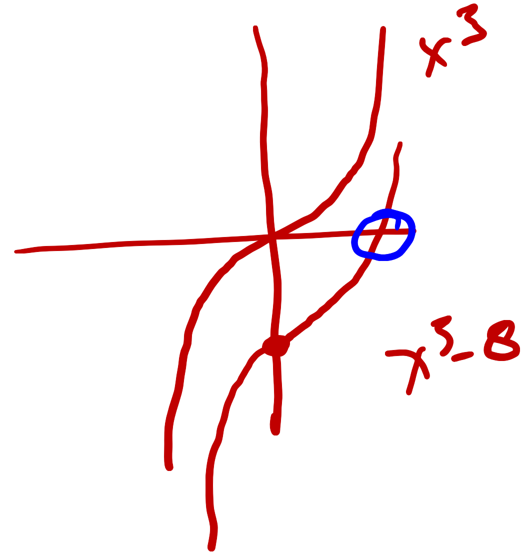
$$\frac{Ax + B}{ax^2 + bx + c}.$$

Evaluate $\int \frac{dx}{x^3-8}$. First factor $x^3 - 8$. $= (x-2)(\cdot \cdot - \; - \;)$

$= (x-2)(x^2+2x+4)$

2 is a root

$$\begin{array}{r|l} & x^2+2x+4 \\ x-2 & x^3 - 8 \\ & x^3-2x^2 \\ \hline & 0 \quad 2x^2-8 \\ & 2x^2-4x \\ \hline & 0+4x-8 \\ & 4x-8 \\ \hline & 0 \end{array}$$



Reducible versus irreducible quadratic:

$x^2 + 2x + 4$

$a = 1 \quad b = 2 \quad c = 4$

$b^2 - 4ac$

$= 4 - 16 = -12$

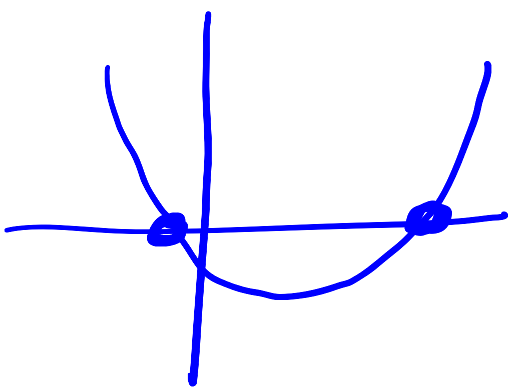
Roots of $ax^2 + bx + c$ are

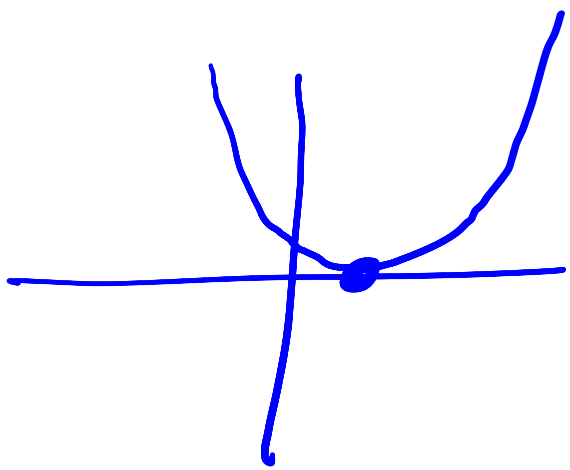
$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

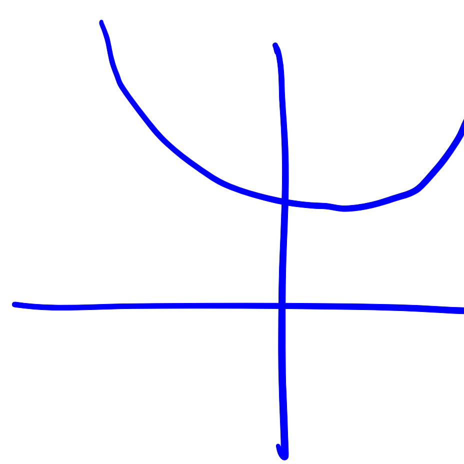
$b^2 - 4ac > 0$ implies 2 roots $\Rightarrow$ reducible. If factor 2 terms

$b^2 - 4ac = 0$ implies 1 root $\Rightarrow$ reducible. factor $(x-r)^2$

$b^2 - 4ac < 0$ implies no roots $\Rightarrow$ irreducible.







Completing the square: An irreducible quadratic can be written as

$$ax^2 + bx + c = A(x - B)^2 + C = A(x^2 - 2Bx + B^2) + C$$

$$\int \frac{1}{x^3-8} = \int \frac{A}{x-2} + \int \frac{Bx+C}{x^2+2x+4}$$

$$1 = A(x^2+2x+4) + (Bx+C)(x-2)$$

$x=2$ $\quad 1 = A(12) + \cancel{(Bx+C)\cdot 0}$

$$1 = A(12)$$

$$\boxed{A = \frac{1}{12}}$$

$x=0$ $\quad 1 = \frac{1}{12}(4) + (B\cdot 0 + C)(-2)$

$$= \frac{1}{3} - 2C$$

$$\frac{2}{3} = -2C \qquad \boxed{C = -1/3}$$

$x=1 \quad 1 = \frac{1}{12}(7) + \left(B - \frac{1}{3}\right)(-1)$

$\frac{12}{12} - \frac{7}{12} - \frac{4}{12} = -B$

$\frac{1}{12} = -B \qquad \boxed{B = \frac{1}{12}}$

$\int \frac{1}{x^3-8}\,dx = \int \frac{1}{12}\frac{1}{x-2} + \int \frac{-\frac{x}{12} - \frac{1}{3}}{x^2+2x+4}\,dx$

$= \frac{1}{12}\int \frac{1}{x-2} - \frac{1}{12}\int \frac{x+4}{x^2+2x+4}\,dx$

$1^{ST} = \frac{1}{12}\ln|x-2|$

Do 2^{nd} complete the square

$\underbrace{x^2+2x+4} = x^2+2x+1-1+4 = (x+1)^2+3$

$\int \frac{x+4}{(x+1)^2+3} = \int \frac{u+3}{u^2+3} = \int \frac{u}{u^2+3} + \int \frac{3}{u^2+3} \qquad a=\sqrt{3}$

$u = x+1 \qquad = \frac{1}{2}\ln(u^2+3) + 3\,\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)$

$= \frac{1}{2}\ln((x+1)^2+3) + \sqrt{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C$

Partial fractions with repeated quadratic term: If Q contains a power of a irreducible quadratic factor $(ax^2+bx+c)^n$ (no real roots), then the partial fraction expansion contains factors of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}.$$

Evaluate $\int \frac{x^3}{(x^2+1)^2}dx$.

$$\frac{x^3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

$$x^3 = (Ax+B)(x^2+1) + Cx+D$$

$$= Ax^3 + Ax + Bx^2 + B + Cx + D$$

$\boxed{1 = A}$ x^3 $\quad \boxed{0 = B}$ x^2 $\quad \boxed{0 = B + D}$ 1 $\quad 0 = A + C$ $\boxed{C = -1}$

$$\int \frac{x^3}{(x^2+1)^2} = \int \frac{1\cdot x + 0}{x^2+1} + \int \frac{-x + 0}{(x^2+1)^2}$$

$$= \int \frac{x}{x^2+1} - \int \frac{x}{(x^2+1)^2}$$

$$= \frac{1}{2} \ln(x^2+1) + \frac{1}{2}(x^2+1)^{-1} + C$$