MAT 126.01, Prof. Bishop, Tuesday, Nov 3, 2020
Recorded lecture
Finish Section 3.3: Trigonometric substitution
Section 3.4: Partial fractions

Section 3.3: Integration involving $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$.

Some such integrals we can already do:

$$\int \frac{dx}{\sqrt{a^2-x^2}}, \qquad \int \frac{x\,dx}{\sqrt{a^2-x^2}}, \qquad \int x\sqrt{a^2-x^2}\,dx$$

In general, for integrals involving:

$\sqrt{a^2 - x^2}$ use $x = a\sin\theta$, $dx = a\cos\theta d\theta$.

$\sqrt{a^2 + x^2}$ use $x = a\tan\theta$, $dx = a\sec^2\theta d\theta$.

$\sqrt{x^2 - a^2}$ use $x = a\sec\theta$, $dx = a\sec\theta\tan\theta d\theta$.

Last week (Oct 29) we did the first case.

Integrals involving $\sqrt{a^2 + x^2}$, use substitution $x = a \tan \theta$.

Evaluate $\int \frac{dx}{\sqrt{1+x^2}}$

Integrals involving $\sqrt{x^2 - a^2}$, use substitution $x = a\sec\theta$.

Evaluate $\int \sqrt{x^2 - 9}dx$

Section 3.4 Partial fractions:

Polynomial functions:

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

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A rational function is a ratio of polynomials:

$$R(x) = \frac{P(x)}{Q(x)}.$$

Using long division we can always write a rational function as a polynomial plus a rational function where the numerator has smaller degree than the denominator.

Long division of numbers

Long division of polynomials

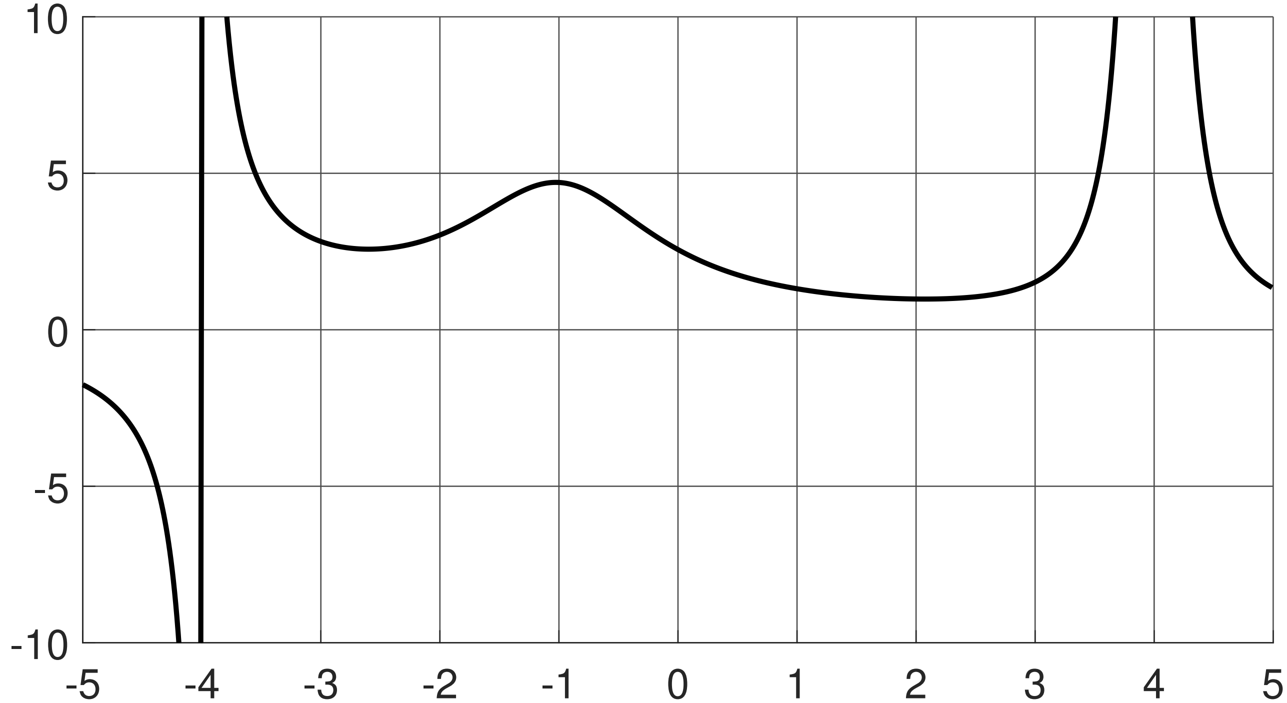
Simplify $\frac{x^2+3x+5}{x+1}$.

Fundamental theorem of algebra: Any polynomial Q with real coefficients can be factored into linear and quadratic factors:

$$Q(x) = (a_1x + b_1)\cdots(a_kx - b_k)(c_1x^2 + d_1x + e_1)\cdots(c_jx^2 + d_jx + e_j).$$

Partial fraction expansion (no repeated factors):

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x + b_1)} + \cdots + \frac{A_k}{(a_kx + b_k)}$$
$$+\frac{C_1x + D_1}{c_1x^2 + d_1x + e_1} + \cdots + \frac{C_jx + D_j}{c_jx^2 + d_jx + e_j}.$$



$$\frac{1}{x+4} \quad + \quad \frac{1}{(x+1)^2+1} \quad + \quad \frac{1}{(x-4)^2}$$

To integrate a rational function $R = P/Q$ we usually want to factor Q and use partial fractions.

By long division we can assume $\deg(P) < \deg(Q)$.

Easiest case is when Q has only linear factors and no repeats. If

$$Q(x) = (a_1 x + b_1) \cdots (a_k x + b_k),$$

then we can write

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \cdots + \frac{A_k}{a_k x + b_k}$$

Evaluate $\int \frac{3x+2}{x^3-x^2-2x}dx$.

Step 1: factor the denominator.

Step 2: solve for partial fraction coefficients.

First method: equate coefficients of powers of x:

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}$$

$$3x+2 = A(x-2)(x+1) + Bx(x+1) + Cx(x-2)$$

$$3x+2 = A(x^2-x-2) + B(x^2+x) + C(x^2-2x)$$

2nd method: strategic substitution. Set x = roots of factors:

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}$$

$$3x+2 = A(x-2)(x+1) + Bx(x+1) + Cx(x-2)$$

Step 3: integrate each term.

Partial fractions with repeated linear factors:

$$\frac{P(x)}{(ax+b)^n} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} \cdots + \frac{A_n}{(ax+b)^n}.$$

Evaluate $\int \frac{x-2}{(2x-1)^2(x-1)} dx$.

Partial fractions with quadratic term: If Q contains a non-repeated irreducible quadratic factor $ax^2 + bx + c$ (no real roots), then the partial fraction expansion contains a factor of the form

$$\frac{Ax + B}{ax^2 + bx + c}.$$

Evaluate $\int \frac{dx}{x^3-8}$. First factor $x^3 - 8$.

Reducible versus irreducible quadratic:

Roots of $ax^2 + bx + c$ are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$b^2 - 4ac > 0$ implies 2 roots $\Rightarrow$ reducible.

$b^2 - 4ac = 0$ implies 1 root $\Rightarrow$ reducible.

$b^2 - 4ac < 0$ implies no roots $\Rightarrow$ irreducible.

Completing the square: An irreducible quadratic can be written as

$$ax^2 + bx + c = A(x - B)^2 + C = A(x^2 - 2Bx + B^2) + C$$

Partial fractions with repeated quadratic term: If Q contains a power of a irreducible quadratic factor $(ax^2 + bx + c)^n$ (no real roots), then the partial fraction expansion contains factors of the form

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}.$$

Evaluate $\int \frac{x^3}{(x^2+1)^2} dx$.