We adopt the following conventions in the integral tables:

1. A constant of integration must be included with all inde nite integrals.
2. All angles are measured in radians; inverse trigonometric and hyperbolic functions represent principal values.
3. Logarithmic expressions are to base $e = 2.71828\ldots$, unless otherwise speci ed, and are to be evaluated for the absolute value of the arguments involved therein.
4. The natural logarithm function is denoted as $\log x$.
5. The variables n and m usually denote integers. The denominator of the expressions shown is not allowed to be zero; this may require that $a \neq 0$ or $m \neq n$ or some other similar statement.
6. When inverse trigonometric functions occur in the integrals, be sure that any replacements made for them are strictly in accordance with the rules for such functions. This causes little dif culty when the argument of the inverse trigonometric function is positive, because all angles involved are in the rst quadrant. However, if the argument is negative, special care must be used. Thus, if $u > 0$ then
$$\sin^{-1} u = \cos^{-1} \sqrt{1-u^2} = \csc^{-1} \frac{1}{u} = \ldots.$$
However, if $u < 0$, then
$$\sin^{-1} u = -\cos^{-1} \sqrt{1-u^2} = -\pi - \csc^{-1} \frac{1}{u} = \ldots.$$

5.4 TABLE OF INDEFINITE INTEGRALS

5.4.1 ELEMENTARY FORMS

1. $\displaystyle\int a\,dx = ax.$

2. $\displaystyle\int a\,f(x)\,dx = a\int f(x)\,dx.$

3. $\displaystyle\int \phi(y(x))\,dx = \int \frac{\phi(y)}{y'}\,dy,$ where $y' = \frac{dy}{dx}$.

4. $\displaystyle\int (u+v)\,dx = \int u\,dx + \int v\,dx,$ where u and v are any functions of x.

5. $\displaystyle\int u\,dv = u\int dv - \int v\,du = uv - \int v\,du.$

6. $\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$

7. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1}$, except when $n = -1$.

8. $\displaystyle\int \frac{dx}{x} = \log x.$

9. $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log f(x),\ \ (df(x) = f'(x)\,dx).$

10. $\displaystyle\int \frac{f'(x)}{2\sqrt{f(x)}}\,dx = \sqrt{f(x)},\ \ (df(x) = f'(x)\,dx).$

11. $\displaystyle\int e^x\,dx = e^x.$

12. $\displaystyle\int e^{ax}\,dx = \frac{e^{ax}}{a}.$

13. $\displaystyle\int b^{ax}\,dx = \frac{b^{ax}}{a\log b},\quad b > 0.$

14. $\displaystyle\int \log x\,dx = x\log x - x.$

15. $\displaystyle\int a^x\,dx = \frac{a^x}{\log a},\quad a > 0.$

16. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}.$

17. $\displaystyle\int \frac{dx}{a^2-x^2} = \begin{cases} \dfrac{1}{a}\tanh^{-1}\dfrac{x}{a}, \\ \text{or} \\ \dfrac{1}{2a}\log\dfrac{a+x}{a-x}, \qquad a^2 > x^2. \end{cases}$

18. $\displaystyle\int \frac{dx}{x^2-a^2} = \begin{cases} -\dfrac{1}{a}\coth^{-1}\dfrac{x}{a}, \\ \text{or} \\ \dfrac{1}{2a}\log\dfrac{x-a}{x+a}, \qquad x^2 > a^2. \end{cases}$

19. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \begin{cases} \sin^{-1}\dfrac{x}{|a|}, \\ \text{or} \\ -\cos^{-1}\dfrac{x}{|a|}, \qquad a^2 > x^2. \end{cases}$

20. $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log\left(x + \sqrt{x^2 \pm a^2}\right).$

21. $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{|a|}\sec^{-1}\frac{x}{a}.$

22. $\displaystyle\int \frac{dx}{x\sqrt{a^2 \pm x^2}} = -\frac{1}{a}\log\left(\frac{a+\sqrt{a^2 \pm x^2}}{x}\right).$

5.4.2 FORMS CONTAINING $a + bx$

23. $\displaystyle\int (a+bx)^n\,dx = \frac{(a+bx)^{n+1}}{(n+1)b},\quad n \neq -1.$

24. $\displaystyle\int x(a+bx)^n\,dx = \frac{1}{b^2(n+2)}(a+bx)^{n+2} - \frac{a}{b^2(n+1)}(a+bx)^{n+1},$
$n \neq -1,\ n \neq -2.$

25. $\int x^2(a+bx)^n\,dx = \frac{1}{b^3}\left[\frac{(a+bx)^{n+3}}{n+3} - 2a\frac{(a+bx)^{n+2}}{n+2} + a^2\frac{(a+bx)^{n+1}}{n+1}\right],$
$n \neq -1, \quad n \neq -2, \; n \neq -3.$

26. $\int x^m(a+bx)^n\,dx =$
$$\begin{cases} \dfrac{x^{m+1}(a+bx)^n}{m+n+1} + \dfrac{an}{m+n+1}\displaystyle\int x^m(a+bx)^{n-1}\,dx, \\ \quad \text{or} \\ \dfrac{1}{a(n+1)}\left[-x^{m+1}(a+bx)^{n+1} + (m+n+2)\displaystyle\int x^m(a+bx)^{n+1}\,dx\right], \\ \quad \text{or} \\ \dfrac{1}{b(m+n+1)}\left[x^m(a+bx)^{n+1} - ma\displaystyle\int x^{m-1}(a+bx)^n\,dx\right]. \end{cases}$$

27. $\int \frac{dx}{a+bx} = \frac{1}{b}\log|a+bx|.$

28. $\int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)}.$

29. $\int \frac{dx}{(a+bx)^3} = -\frac{1}{2b(a+bx)^2}.$

30. $\int \frac{x}{a+bx}\,dx = \begin{cases} \dfrac{1}{b^2}\left[a+bx-a\log(a+bx)\right], \\ \quad \text{or} \\ \dfrac{x}{b} - \dfrac{a}{b^2}\log(a+bx). \end{cases}$

31. $\int \frac{x}{(a+bx)^2}\,dx = \frac{1}{b^2}\left[\log(a+bx) + \frac{a}{a+bx}\right].$

32. $\int \frac{x}{(a+bx)^n}\,dx = \frac{1}{b^2}\left[\frac{-1}{(n-2)(a+bx)^{n-2}} + \frac{a}{(n-1)(a+bx)^{n-1}}\right],$
$n \neq 1, \; n \neq 2.$

33. $\int \frac{x^2}{a+bx}\,dx = \frac{1}{b^3}\left(\frac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2\log(a+bx)\right).$

34. $\int \frac{x^2}{(a+bx)^2}\,dx = \frac{1}{b^3}\left(a+bx-2a\log(a+bx) - \frac{a^2}{a+bx}\right).$

35. $\int \frac{x^2}{(a+bx)^3}\,dx = \frac{1}{b^3}\left(\log(a+bx) + \frac{2a}{a+bx} - \frac{a^2}{2(a+bx)^2}\right).$

36. $\int \frac{x^2}{(a+bx)^n}\,dx = \frac{1}{b^3}\left[\frac{-1}{(n-3)(a+bx)^{n-3}} + \frac{2a}{(n-2)(a+bx)^{n-2}} - \frac{a^2}{(n-1)(a+bx)^{n-1}}\right], \quad n \neq 1, \; n \neq 2, \; n \neq 3.$

37. $\int \frac{dx}{x(a+bx)} = -\frac{1}{a}\log\frac{a+bx}{x}.$

38. $\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2}\log\frac{a+bx}{x}.$

39. $\int \frac{dx}{x(a+bx)^3} = \frac{1}{a^3}\left[\frac{1}{2}\left(\frac{2a+bx}{a+bx}\right)^2 - \log\frac{a+bx}{x}\right].$

40. $\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\log\frac{a+bx}{x}.$

41. $\displaystyle\int \frac{dx}{x^3(a+bx)} = \frac{2bx-a}{2a^2x^2} + \frac{b^2}{a^3}\log\frac{x}{a+bx}.$

42. $\displaystyle\int \frac{dx}{x^2(a+bx)^2} = -\frac{a+2bx}{a^2x(a+bx)} + \frac{2b}{a^3}\log\frac{a+bx}{x}.$

5.4.3 FORMS CONTAINING $c^2 \pm x^2$ AND $x^2 - c^2$

43. $\displaystyle\int \frac{dx}{c^2+x^2} = \frac{1}{c}\tan^{-1}\frac{x}{c}.$

44. $\displaystyle\int \frac{dx}{c^2-x^2} = \frac{1}{2c}\log\frac{c+x}{c-x}, \quad c^2 > x^2.$

45. $\displaystyle\int \frac{dx}{x^2-c^2} = \frac{1}{2c}\log\frac{x-c}{x+c}, \quad x^2 > c^2.$

46. $\displaystyle\int \frac{x}{c^2 \pm x^2}\,dx = \pm\frac{1}{2}\log\left(c^2 \pm x^2\right).$

47. $\displaystyle\int \frac{x}{(c^2 \pm x^2)^{n+1}}\,dx = \mp\frac{1}{2n(c^2 \pm x^2)^n}, \quad n \neq 0.$

48. $\displaystyle\int \frac{dx}{(c^2 \pm x^2)^n} = \frac{1}{2c^2(n-1)}\left[\frac{x}{(c^2 \pm x^2)^{n-1}} + (2n-3)\int\frac{dx}{(c^2 \pm x^2)^{n-1}}\right].$

49. $\displaystyle\int \frac{dx}{(x^2 - c^2)^n} = \frac{1}{2c^2(n-1)}\left[-\frac{x}{(x^2 - c^2)^{n-1}} - (2n-3)\int\frac{dx}{(x^2 - c^2)^{n-1}}\right].$

50. $\displaystyle\int \frac{x}{x^2 - c^2}\,dx = \frac{1}{2}\log\left(x^2 - c^2\right).$

51. $\displaystyle\int \frac{x}{(x^2 - c^2)^{n+1}}\,dx = -\frac{1}{2n(x^2 - c^2)^n}.$

5.4.4 FORMS CONTAINING $a + bx$ AND $c + dx$

$u = a + bx, \quad v = c + dx,$ and $k = ad - bc.$ (If $k = 0$, then $v = (c/a)u$.)

52. $\displaystyle\int \frac{dx}{uv} = \frac{1}{k}\log\left(\frac{v}{u}\right).$

53. $\displaystyle\int \frac{x}{uv}\,dx = \frac{1}{k}\left(\frac{a}{b}\log u - \frac{c}{d}\log v\right).$

54. $\displaystyle\int \frac{dx}{u^2v} = \frac{1}{k}\left(\frac{1}{u} + \frac{d}{k}\log\frac{v}{u}\right).$

55. $\displaystyle\int \frac{x}{u^2v}\,dx = -\frac{a}{bku} - \frac{c}{k^2}\log\frac{v}{u}.$

56. $\displaystyle\int \frac{x^2}{u^2v}\,dx = \frac{a^2}{b^2ku} + \frac{1}{k^2}\left(\frac{c^2}{d}\log v + \frac{a(k-bc)}{b^2}\log u\right).$

57. $\displaystyle\int \frac{dx}{u^nv^m} = \frac{1}{k(m-1)}\left[\frac{-1}{u^{n-1}v^{m-1}} - b(m+n-2)\int\frac{dx}{u^nv^{m-1}}\right].$

58. $\displaystyle\int \frac{u}{v}\,dx = \frac{bx}{d} + \frac{k}{d^2}\log v.$

59. $$\int \frac{u^m}{v^n}\,dx = \begin{cases} -\dfrac{1}{k(n-1)}\left[\dfrac{u^{m+1}}{v^{n-1}} + b(n-m-2)\displaystyle\int \frac{u^m}{v^{n-1}}\,dx\right], \\ \text{or} \\ -\dfrac{1}{d(n-m-1)}\left[\dfrac{u^m}{v^{n-1}} + mk\displaystyle\int \frac{u^{m-1}}{v^n}\,dx\right], \\ \text{or} \\ -\dfrac{1}{d(n-1)}\left[\dfrac{u^m}{v^{n-1}} - mb\displaystyle\int \frac{u^{m-1}}{v^{n-1}}\,dx\right]. \end{cases}$$

5.4.5 FORMS CONTAINING $a + bx^n$

60. $$\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}}\tan^{-1}\frac{x\sqrt{ab}}{a}, \quad ab > 0.$$

61. $$\int \frac{dx}{a+bx^2} = \begin{cases} \dfrac{1}{2\sqrt{-ab}}\log\dfrac{a+x\sqrt{-ab}}{a-x\sqrt{-ab}}, & ab < 0, \\ \text{or} \\ \dfrac{1}{\sqrt{-ab}}\tanh^{-1}\dfrac{x\sqrt{-ab}}{a}, & ab < 0. \end{cases}$$

62. $$\int \frac{dx}{a^2+b^2x^2}\,dx = \frac{1}{ab}\tan^{-1}\frac{bx}{a}.$$

63. $$\int \frac{x}{a+bx^2}\,dx = \frac{1}{2b}\log\left(a+bx^2\right).$$

64. $$\int \frac{x^2}{a+bx^2}\,dx = \frac{x}{b} - \frac{a}{b}\int \frac{dx}{a+bx^2}.$$

65. $$\int \frac{dx}{(a+bx^2)^2} = \frac{x}{2a(a+bx^2)} + \frac{1}{2a}\int \frac{dx}{a+bx^2}.$$

66. $$\int \frac{dx}{a^2-b^2x^2} = \frac{1}{2ab}\log\frac{a+bx}{a-bx}.$$

67. $$\int \frac{dx}{(a+bx^2)^{m+1}} =$$
$$\begin{cases} \dfrac{1}{2ma}\dfrac{x}{(a+bx^2)^m} + \dfrac{2m-1}{2ma}\displaystyle\int \frac{dx}{(a+bx^2)^m}, \\ \text{or} \\ \dfrac{(2m)!}{(m!)^2}\left[\dfrac{x}{2a}\displaystyle\sum_{r=1}^{m}\frac{r!(r-1)!}{(4a)^{m-r}(2r)!(a+bx^2)^r} + \frac{1}{(4a)^m}\int \frac{dx}{a+bx^2}\right]. \end{cases}$$

68. $$\int \frac{x\,dx}{(a+bx^2)^{m+1}} = -\frac{1}{2bm(a+bx^2)^m}, \quad m \neq 0.$$

69. $$\int \frac{x^2\,dx}{(a+bx^2)^{m+1}} = -\frac{x}{2mb(a+bx^2)^m} + \frac{1}{2mb}\int \frac{dx}{(a+bx^2)^m}, \quad m \neq 0.$$

70. $$\int \frac{dx}{x(a+bx^2)} = \frac{1}{2a}\log\frac{x^2}{a+bx^2}.$$

71. $$\int \frac{dx}{x^2(a+bx^2)} = -\frac{1}{ax} - \frac{b}{a}\int \frac{dx}{a+bx^2}.$$

72. $$\int \frac{dx}{x\left(a+bx^2\right)^{m+1}} = \begin{cases} \dfrac{1}{2am(a+bx^2)^m} + \dfrac{1}{a}\displaystyle\int \frac{dx}{x\left(a+bx^2\right)^m}, \\ \text{or} \\ \dfrac{1}{2a^{m+1}}\left[\displaystyle\sum_{r=1}^{m}\frac{a^r}{r(a+bx^2)^r} + \log\frac{x^2}{a+bx^2}\right]. \end{cases}$$

73. $\displaystyle\int \frac{dx}{x^2\,(a+bx^2)^{m+1}} = \frac{1}{a}\int \frac{dx}{x^2(a+bx^2)^m} - \frac{b}{a}\int \frac{dx}{(a+bx^2)^{m+1}}.$

74. $\displaystyle\int \frac{dx}{a+bx^3} = \frac{k}{3a}\left[\frac{1}{2}\log\frac{(k+x)^3}{a+bx^3} + \sqrt{3}\tan^{-1}\frac{2x-k}{k\sqrt{3}}\right],\quad k = \sqrt[3]{\tfrac{a}{b}}.$

75. $\displaystyle\int \frac{x\,dx}{a+bx^3} = \frac{1}{3bk}\left[\frac{1}{2}\log\frac{a+bx^3}{(k+x)^3} + \sqrt{3}\tan^{-1}\frac{2x-k}{k\sqrt{3}}\right],\quad k = \sqrt[3]{\tfrac{a}{b}}.$

76. $\displaystyle\int \frac{x^2\,dx}{a+bx^3} = \frac{1}{3b}\log a+bx^3.$

77. $\displaystyle\int \frac{dx}{a+bx^4} =$

$$\begin{cases} \dfrac{k}{2a}\left[\dfrac{1}{2}\log\dfrac{x^2+2kx+2k^2}{x^2-2kx+2k^2} + \tan^{-1}\dfrac{2kx}{2k^2-x^2}\right], & ab>0,\ k=\left(\dfrac{a}{4b}\right)^{1/4}, \\ \text{or} & \\ \dfrac{k}{2a}\left[\dfrac{1}{2}\log\dfrac{x+k}{x-k} + \tan^{-1}\dfrac{x}{k}\right], & ab<0,\ k=\left(-\dfrac{a}{b}\right)^{1/4}. \end{cases}$$

78. $\displaystyle\int \frac{x}{a+bx^4}\,dx = \frac{1}{2bk}\tan^{-1}\frac{x^2}{k},\quad ab>0,\ k=\sqrt{\tfrac{a}{b}}.$

79. $\displaystyle\int \frac{x}{a+bx^4}\,dx = \frac{1}{4bk}\log\frac{x^2-k}{x^2+k},\quad ab<0,\ k=\sqrt{-\tfrac{a}{b}}.$

80. $\displaystyle\int \frac{x^2}{a+bx^4}\,dx = \frac{1}{4bk}\left[\frac{1}{2}\log\frac{x^2-2kx+2k^2}{x^2+2kx+2k^2} + \tan^{-1}\frac{2kx}{2k^2-x^2}\right],$

$ab>0,\ k=\left(\tfrac{a}{4b}\right)^{1/4}.$

81. $\displaystyle\int \frac{x^2\,dx}{a+bx^4} = \frac{1}{4bk}\left[\log\frac{x-k}{x+k} + 2\tan^{-1}\frac{x}{k}\right],\quad ab<0,\ k=\sqrt[4]{-\tfrac{a}{b}}.$

82. $\displaystyle\int \frac{x^3\,dx}{a+bx^4} = \frac{1}{4b}\log\left(a+bx^4\right).$

83. $\displaystyle\int \frac{dx}{x(a+bx^n)} = \frac{1}{an}\log\frac{x^n}{a+bx^n},\quad n\neq 0.$

84. $\displaystyle\int \frac{dx}{(a+bx^n)^{m+1}} = \frac{1}{a}\int \frac{dx}{(a+bx^n)^m} - \frac{b}{a}\int \frac{x^n\,dx}{(a+bx^n)^{m+1}}.$

85. $\displaystyle\int \frac{x^m\,dx}{(a+bx^n)^{p+1}} = \frac{1}{b}\int \frac{x^{m-n}\,dx}{(a+bx^n)^p} - \frac{a}{b}\int \frac{x^{m-n}\,dx}{(a+bx^n)^{p+1}}.$

86. $\displaystyle\int \frac{dx}{x^m(a+bx^n)^{p+1}} = \frac{1}{a}\int \frac{dx}{x^m(a+bx^n)^p} - \frac{b}{a}\int \frac{dx}{x^{m-n}(a+bx^n)^{p+1}}.$

87. $\displaystyle\int x^m(a+bx^n)^p\,dx =$

$$\begin{cases} \dfrac{1}{b(np+m+1)}\left[x^{m-n+1}(a+bx^n)^{p+1} - a(m-n+1)\displaystyle\int x^{m-n}(a+bx^n)^p\,dx\right], \\ \text{or} \\ \dfrac{1}{np+m+1}\left[x^{m+1}(a+bx^n)^p + anp\displaystyle\int x^m(a+bx^n)^{p-1}\,dx\right], \\ \text{or} \\ \dfrac{1}{a(m+1)}\left[x^{m+1}(a+bx^n)^{p+1} - b(m+1+np+n)\displaystyle\int x^{m+n}(a+bx^n)^p\,dx\right], \\ \text{or} \\ \dfrac{1}{an(p+1)}\left[-x^{m+1}(a+bx^n)^{p+1} + (m+1+np+n)\displaystyle\int x^m(a+bx^n)^{p+1}\,dx\right]. \end{cases}$$

5.4.6 FORMS CONTAINING $c^3 \pm x^3$

88. $\displaystyle\int \frac{dx}{c^3 \pm x^3} = \pm\frac{1}{6c^2}\log\left(\frac{(c\pm x)^3}{c^3 \pm x^3}\right) + \frac{1}{c^2\sqrt{3}}\tan^{-1}\frac{2x \mp c}{c\sqrt{3}}.$

89. $\displaystyle\int \frac{dx}{(c^3 \pm x^3)^2} = \frac{x}{3c^3(c^3 \pm x^3)} + \frac{2}{3c^3}\int \frac{dx}{c^3 \pm x^3}.$

90. $\displaystyle\int \frac{dx}{(c^3 \pm x^3)^{n+1}} = \frac{1}{3nc^3}\left[\frac{x}{(c^3 \pm x^3)^n} + (3n-1)\int \frac{dx}{(c^3 \pm x^3)^n}\right], \quad n \neq 0.$

91. $\displaystyle\int \frac{x\,dx}{c^3 \pm x^3} = \frac{1}{6c}\log\frac{c^3 \pm x^3}{(c\pm x)^3} \pm \frac{1}{c\sqrt{3}}\tan^{-1}\frac{2x \mp c}{c\sqrt{3}}.$

92. $\displaystyle\int \frac{x\,dx}{(c^3 \pm x^3)^2} = \frac{x^2}{3c^3(c^3 \pm x^3)} + \frac{1}{3c^3}\int \frac{x\,dx}{c^3 \pm x^3}.$

93. $\displaystyle\int \frac{x\,dx}{(c^3 \pm x^3)^{n+1}} = \frac{1}{3nc^3}\left[\frac{x^2}{(c^3 \pm x^3)^n} + (3n-2)\int \frac{x\,dx}{(c^3 \pm x^3)^n}\right], \quad n \neq 0.$

94. $\displaystyle\int \frac{x^2\,dx}{c^3 \pm x^3} = \pm\frac{1}{3}\log\left(c^3 \pm x^3\right).$

95. $\displaystyle\int \frac{x^2\,dx}{(c^3 \pm x^3)^{n+1}} = \mp\frac{1}{3n(c^3 \pm x^3)^n}, \quad n \neq 0.$

96. $\displaystyle\int \frac{dx}{x(c^3 \pm x^3)} = \frac{1}{3c^3}\log\frac{x^3}{c^3 \pm x^3}.$

97. $\displaystyle\int \frac{dx}{x(c^3 \pm x^3)^2} = \frac{1}{3c^3(c^3 \pm x^3)} + \frac{1}{3c^6}\log\frac{x^3}{c^3 \pm x^3}.$

98. $\displaystyle\int \frac{dx}{x(c^3 \pm x^3)^{n+1}} = \frac{1}{3nc^3(c^3 \pm x^3)^n} + \frac{1}{c^3}\int \frac{dx}{x(c^3 \pm x^3)^n}, \quad n \neq 0.$

99. $\displaystyle\int \frac{dx}{x^2(c^3 \pm x^3)} = -\frac{1}{c^3 x} \mp \frac{1}{c^3}\int \frac{x\,dx}{(c^3 \pm x^3)}.$

100. $\displaystyle\int \frac{dx}{x^2(c^3 \pm x^3)^{n+1}} = \frac{1}{c^3}\int \frac{dx}{x^2(c^3 \pm x^3)^n} \mp \frac{1}{c^3}\int \frac{x\,dx}{(c^3 \pm x^3)^{n+1}}.$

5.4.7 FORMS CONTAINING $c^4 \pm x^4$

101. $\displaystyle\int \frac{dx}{c^4 + x^4} = \frac{1}{2c^3\sqrt{2}}\left[\frac{1}{2}\log\left(\frac{x^2 + cx\sqrt{2} + c^2}{x^2 - cx\sqrt{2} + c^2}\right) + \tan^{-1}\frac{cx\sqrt{2}}{c^2 - x^2}\right].$

102. $\displaystyle\int \frac{dx}{c^4 - x^4} = \frac{1}{2c^3}\left[\frac{1}{2}\log\frac{c+x}{c-x} + \tan^{-1}\frac{x}{c}\right].$

103. $\displaystyle\int \frac{x\,dx}{c^4 + x^4} = \frac{1}{2c^2}\tan^{-1}\frac{x^2}{c^2}.$

104. $\displaystyle\int \frac{x\,dx}{c^4 - x^4} = \frac{1}{4c^2}\log\frac{c^2 + x^2}{c^2 - x^2}.$

105. $\displaystyle\int \frac{x^2\,dx}{c^4 + x^4} = \frac{1}{2c\sqrt{2}}\left[\frac{1}{2}\log\left(\frac{x^2 - cx\sqrt{2} + c^2}{x^2 + cx\sqrt{2} + c^2}\right) + \tan^{-1}\frac{cx\sqrt{2}}{c^2 - x^2}\right].$

106. $\displaystyle\int \frac{x^2\,dx}{c^4 - x^4} = \frac{1}{2c}\left[\frac{1}{2}\log\frac{c+x}{c-x} - \tan^{-1}\frac{x}{c}\right].$

107. $\displaystyle\int \frac{x^3\,dx}{c^4 \pm x^4} = \pm\frac{1}{4}\log\left(c^4 \pm x^4\right).$

5.4.8 FORMS CONTAINING $a + bx + cx^2$

$$X = a + bx + cx^2 \quad \text{and} \quad q = 4ac - b^2.$$

If $q = 0$, then $X = c\left(x + \frac{b}{2c}\right)^2$ and other formulae should be used.

108. $$\int \frac{dx}{X} = \begin{cases} \dfrac{2}{\sqrt{q}}\tan^{-1}\dfrac{2cx+b}{\sqrt{q}}, & q > 0, \\ \text{or} & \\ \dfrac{-2}{\sqrt{-q}}\tanh^{-1}\dfrac{2cx+b}{\sqrt{-q}}, & q < 0, \\ \text{or} & \\ \dfrac{1}{\sqrt{-q}}\log\dfrac{2cx+b-\sqrt{-q}}{2cx+b+\sqrt{-q}}, & q < 0. \end{cases}$$

109. $\displaystyle\int \frac{dx}{X^2} = \frac{2cx+b}{qX} + \frac{2c}{q}\int\frac{dx}{X}.$

110. $\displaystyle\int \frac{dx}{X^3} = \frac{2cx+b}{q}\left(\frac{1}{2X^2} + \frac{3c}{qX}\right) + \frac{6c^2}{q^2}\int\frac{dx}{X}.$

111. $$\int \frac{dx}{X^{n+1}} = \begin{cases} \dfrac{2cx+b}{nqX^n} + \dfrac{2(2n-1)c}{qn}\displaystyle\int\frac{dx}{X^n}, \\ \text{or} \\ \dfrac{(2n)!}{(n!)^2}\left(\dfrac{c}{q}\right)^n\left[\dfrac{2cx+b}{q}\displaystyle\sum_{r=1}^{n}\left(\dfrac{q}{cX}\right)^r\left(\dfrac{(r-1)!r!}{(2r)!}\right) + \displaystyle\int\frac{dx}{X}\right]. \end{cases}$$

112. $\displaystyle\int \frac{x\,dx}{X} = \frac{1}{2c}\log X - \frac{b}{2c}\int\frac{dx}{X}.$

113. $\displaystyle\int \frac{x\,dx}{X^2} = -\frac{bx+2a}{qX} - \frac{b}{q}\int\frac{dx}{X}.$

114. $\displaystyle\int \frac{x\,dx}{X^{n+1}} = -\frac{2a+bx}{nqX^n} - \frac{b(2n-1)}{nq}\int\frac{dx}{X^n}, \quad n \neq 0.$

115. $\displaystyle\int \frac{x^2\,dx}{X} = \frac{x}{c} - \frac{b}{2c^2}\log X + \frac{b^2-2ac}{2c^2}\int\frac{dx}{X}.$

116. $\displaystyle\int \frac{x^2\,dx}{X^2} = \frac{(b^2-2ac)x+ab}{cqX} + \frac{2a}{q}\int\frac{dx}{X}.$

117. $$\int \frac{x^m\,dx}{X^{n+1}} = -\frac{x^{m-1}}{(2n-m+1)cX^n} - \frac{n-m+1}{2n-m+1}\frac{b}{c}\int\frac{x^{m-1}}{X^{n+1}}\,dx + \frac{m-1}{2n-m+1}\frac{a}{c}\int\frac{x^{m-2}}{X^{n+1}}\,dx.$$

118. $\displaystyle\int \frac{dx}{xX} = \frac{1}{2a}\log\frac{x^2}{X} - \frac{b}{2a}\int\frac{dx}{X}.$

119. $\displaystyle\int \frac{dx}{x^2X} = \frac{b}{2a^2}\log\frac{X}{x^2} - \frac{1}{ax} + \left(\frac{b^2}{2a^2} - \frac{c}{a}\right)\int\frac{dx}{X}.$

120. $\displaystyle\int \frac{dx}{xX^n} = \frac{1}{2a(n-1)X^{n-1}} - \frac{b}{2a}\int\frac{dx}{X^n} + \frac{1}{a}\int\frac{dx}{xX^{n-1}}, \quad n \neq 1.$

121. $$\int \frac{dx}{x^m X^{n+1}} = -\frac{1}{(m-1)ax^{m-1}X^n} - \frac{n+m-1}{m-1}\frac{b}{a}\int \frac{dx}{x^{m-1}X^{n+1}} - \frac{2n+m-1}{m-1}\frac{c}{a}\int \frac{dx}{x^{m-2}X^{n+1}}.$$

5.4.9 FORMS CONTAINING $\sqrt{a+bx}$

122. $$\int \sqrt{a+bx}\,dx = \frac{2}{3b}\sqrt{(a+bx)^3}.$$

123. $$\int x\sqrt{a+bx}\,dx = -\frac{2(2a-3bx)}{15b^2}\sqrt{(a+bx)^3}.$$

124. $$\int x^2\sqrt{a+bx}\,dx = \frac{2(8a^2-12abx+15b^2x^2)}{105b^3}\sqrt{(a+bx)^3}.$$

125. $$\int x^m\sqrt{a+bx}\,dx = \begin{cases} \dfrac{2}{b(2m+3)}\left[x^m\sqrt{(a+bx)^3} - ma\displaystyle\int x^{m-1}\sqrt{a+bx}\,dx\right], \\ \text{or} \\ \dfrac{2}{b^{m+1}}\sqrt{a+bx}\displaystyle\sum_{r=0}^{m}\frac{m!(-a)^{m-r}}{r!(m-r)!(2r+3)}(a+bx)^{r+1}. \end{cases}$$

126. $$\int \frac{\sqrt{a+bx}}{x}\,dx = 2\sqrt{a+bx} + a\int \frac{dx}{x\sqrt{a+bx}}.$$

127. $$\int \frac{\sqrt{a+bx}}{x^m}\,dx = -\frac{1}{(m-1)a}\left[\frac{\sqrt{(a+bx)^3}}{x^{m-1}} + \frac{(2m-5)b}{2}\int \frac{\sqrt{a+bx}}{x^{m-1}}\,dx\right].$$

128. $$\int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b}.$$

129. $$\int \frac{x\,dx}{\sqrt{a+bx}} = -\frac{2(2a-bx)}{3b^2}\sqrt{a+bx}.$$

130. $$\int \frac{x^2\,dx}{\sqrt{a+bx}} = \frac{2(8a^2-4abx+3b^2x^2)}{15b^3}\sqrt{a+bx}.$$

131. $$\int \frac{x^m\,dx}{\sqrt{a+bx}} = \begin{cases} \dfrac{2}{(2m+1)b}\left[x^m\sqrt{a+bx} - ma\displaystyle\int \frac{x^{m-1}}{\sqrt{a+bx}}\,dx\right], \\ \text{or} \\ \dfrac{2(-a)^m\sqrt{a+bx}}{b^{m+1}}\displaystyle\sum_{r=0}^{m}\frac{(-1)^r m!(a+bx)^r}{(2r+1)r!(m-r)!a^r}. \end{cases}$$

132. $$\int \frac{dx}{x\sqrt{a+bx}} = \begin{cases} \dfrac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\dfrac{a+bx}{-a}}, & a<0, \\ \text{or} & \\ \dfrac{1}{\sqrt{a}}\log\left(\dfrac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right), & a>0. \end{cases}$$

133. $$\int \frac{dx}{x^2\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{a+bx}}.$$

134. $\int \frac{dx}{x^n\sqrt{a+bx}} =$
$$\begin{cases} -\frac{\sqrt{a+bx}}{(n-1)ax^{n-1}} - \frac{(2n-3)b}{(2n-2)a}\int \frac{dx}{x^{n-1}\sqrt{a+bx}}, \\ \text{or} \\ \frac{(2n-2)!}{[(n-1)!]^2}\left[-\frac{\sqrt{a+bx}}{a}\sum_{r=1}^{n-1}\frac{r!(r-1)!}{x^r(2r)!}\left(-\frac{b}{4a}\right)^{n-r-1} + \left(-\frac{b}{4a}\right)^{n-1}\int\frac{dx}{x\sqrt{a+bx}}\right]. \end{cases}$$

135. $\int (a+bx)^{\pm n/2}\,dx = \frac{2(a+bx)^{(2\pm n)/2}}{b(2\pm n)}$.

136. $\int x(a+bx)^{\pm n/2}\,dx = \frac{2}{b^2}\left[\frac{(a+bx)^{(4\pm n)/2}}{4\pm n} - \frac{a(a+bx)^{(2\pm n)/2}}{2\pm n}\right]$.

137. $\int \frac{dx}{x(a+bx)^{n/2}} = \frac{1}{a}\int\frac{dx}{x(a+bx)^{(n-2)/2}} - \frac{b}{a}\int\frac{dx}{(a+bx)^{n/2}}$.

138. $\int \frac{(a+bx)^{n/2}}{x}\,dx = b\int (a+bx)^{(n-2)/2}\,dx + a\int\frac{(a+bx)^{(n-2)/2}}{x}\,dx$.

5.4.10 FORMS CONTAINING $\sqrt{a+bx}$ AND $\sqrt{c+dx}$

and $\sqrt{c+dx}$

$$u = a+bx, \qquad v = c+dx, \qquad k = ad-bc.$$

If $k=0$, then $v = \frac{c}{a}u$, and other formulae should be used.

139. $$\int\frac{dx}{\sqrt{uv}} = \begin{cases} \frac{2}{\sqrt{bd}}\tanh^{-1}\frac{\sqrt{bduv}}{bv}, & bd>0,\ k<0, \\ \frac{2}{\sqrt{bd}}\tanh^{-1}\frac{\sqrt{bduv}}{du}, & bd>0,\ k>0, \\ \frac{1}{\sqrt{bd}}\log\frac{(bv+\sqrt{bduv})^2}{v}, & bd>0, \\ \frac{2}{\sqrt{-bd}}\tan^{-1}\frac{\sqrt{-bduv}}{bv}, & bd<0, \\ -\frac{1}{\sqrt{-bd}}\sin^{-1}\left(\frac{2bdx+ad+bc}{|k|}\right), & bd<0. \end{cases}$$

140. $\int\sqrt{uv}\,dx = \frac{k+2bv}{4bd}\sqrt{uv} - \frac{k^2}{8bd}\int\frac{dx}{\sqrt{uv}}$.

141. $$\int\frac{dx}{v\sqrt{u}} = \begin{cases} \frac{1}{\sqrt{kd}}\log\frac{d\sqrt{u}-\sqrt{kd}}{d\sqrt{u}+\sqrt{kd}}, & kd>0, \\ \text{or} \\ \frac{1}{\sqrt{kd}}\log\frac{\left(d\sqrt{u}-\sqrt{kd}\right)^2}{v}, & kd>0, \\ \text{or} \\ \frac{2}{\sqrt{-kd}}\tan^{-1}\frac{d\sqrt{u}}{\sqrt{-kd}}, & kd<0. \end{cases}$$

142. $\int\frac{x\,dx}{\sqrt{uv}} = \frac{\sqrt{uv}}{bd} - \frac{ad+bc}{2bd}\int\frac{dx}{\sqrt{uv}}$.

143. $\displaystyle\int \frac{dx}{v\sqrt{uv}} = -\frac{2\sqrt{uv}}{kv}.$

144. $\displaystyle\int \frac{v\,dx}{\sqrt{uv}} = \frac{\sqrt{uv}}{b} - \frac{k}{2b}\int \frac{dx}{\sqrt{uv}}.$

145. $\displaystyle\int \sqrt{\frac{v}{u}}\,dx = \frac{v}{|v|}\int \frac{v\,dx}{\sqrt{uv}}.$

146. $\displaystyle\int v^m\sqrt{u}\,dx = \frac{1}{(2m+3)d}\left(2v^{m+1}\sqrt{u} + k\int \frac{v^m\,dx}{\sqrt{u}}\right).$

147. $\displaystyle\int \frac{dx}{v^m\sqrt{u}} = -\frac{1}{(m-1)k}\left(\frac{\sqrt{u}}{v^{m-1}} + \left(m-\frac{3}{2}\right)b\int \frac{dx}{v^{m-1}\sqrt{u}}\right), \quad m \neq 1.$

148. $$\int \frac{v^m}{\sqrt{u}}\,dx = \begin{cases} \dfrac{2}{b(2m+1)}\left(v^m\sqrt{u} - mk\displaystyle\int \frac{v^{m-1}}{\sqrt{u}}\,dx\right), \\ \quad \text{or} \\ \dfrac{2(m!)^2\sqrt{u}}{b(2m+1)!}\displaystyle\sum_{r=0}^{m}\left(-\frac{4k}{b}\right)^{m-r}\frac{(2r)!}{(r!)^2}v^r. \end{cases}$$

5.4.11 FORMS CONTAINING $\sqrt{x^2 \pm a^2}$

149. $\displaystyle\int \sqrt{x^2 \pm a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2 \pm a^2} \pm a^2\log\left(x + \sqrt{x^2 \pm a^2}\right)\right].$

150. $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log\left(x + \sqrt{x^2 \pm a^2}\right).$

151. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{|a|}\sec^{-1}\frac{x}{a}.$

152. $\displaystyle\int \frac{dx}{x\sqrt{x^2 + a^2}} = -\frac{1}{a}\log\left(\frac{a + \sqrt{x^2 + a^2}}{x}\right).$

153. $\displaystyle\int \frac{\sqrt{x^2 + a^2}}{x}\,dx = \sqrt{x^2 + a^2} - a\log\left(\frac{a + \sqrt{x^2 + a^2}}{x}\right).$

154. $\displaystyle\int \frac{\sqrt{x^2 - a^2}}{x}\,dx = \sqrt{x^2 - a^2} - |a|\sec^{-1}\frac{x}{a}.$

155. $\displaystyle\int \frac{x}{\sqrt{x^2 \pm a^2}}\,dx = \sqrt{x^2 \pm a^2}.$

156. $\displaystyle\int x\sqrt{x^2 \pm a^2}\,dx = \frac{1}{3}\sqrt{(x^2 \pm a^2)^3}.$

157. $$\int \sqrt{(x^2 \pm a^2)^3}\,dx = \frac{1}{4}\left[x\sqrt{(x^2 \pm a^2)^3} \pm \frac{3a^2x}{2}\sqrt{x^2 \pm a^2} + \frac{3a^4}{2}\log\left(x + \sqrt{x^2 \pm a^2}\right)\right].$$

158. $\displaystyle\int \frac{dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}.$

159. $\displaystyle\int \frac{x}{\sqrt{(x^2 \pm a^2)^3}}\,dx = \frac{-1}{\sqrt{x^2 \pm a^2}}.$

160. $\displaystyle\int x\sqrt{(x^2 \pm a^2)^3}\,dx = \frac{1}{5}\sqrt{(x^2 \pm a^2)^5}.$

161. $\displaystyle\int x^2\sqrt{x^2 \pm a^2}\,dx = \frac{x}{4}\sqrt{(x^2 \pm a^2)^3} \mp \frac{a^2}{8}x\sqrt{x^2 \pm a^2} - \frac{a^4}{8}\log\left(x + \sqrt{x^2 \pm a^2}\right).$

162. $\int x^3\sqrt{x^2+a^2}\,dx = \frac{1}{15}(3x^2-2a^2)\sqrt{(x^2+a^2)^3}.$

163. $\int x^3\sqrt{x^2-a^2}\,dx = \frac{1}{5}\sqrt{(x^2-a^2)^5} + \frac{a^2}{3}\sqrt{(x^2-a^2)^3}.$

164. $\int \frac{x^2}{\sqrt{x^2\pm a^2}}\,dx = \frac{x}{2}\sqrt{x^2\pm a^2} \mp \frac{a^2}{2}\log\left(x+\sqrt{x^2\pm a^2}\right).$

165. $\int \frac{x^3}{\sqrt{x^2\pm a^2}}\,dx = \frac{1}{3}\sqrt{(x^2\pm a^2)^3} \mp a^2\sqrt{x^2\pm a^2}.$

166. $\int \frac{dx}{x^2\sqrt{x^2\pm a^2}}\,dx = \mp\frac{\sqrt{x^2\pm a^2}}{a^2x}.$

167. $\int \frac{dx}{x^3\sqrt{x^2+a^2}}\,dx = -\frac{\sqrt{x^2+a^2}}{2a^2x^2} + \frac{1}{2a^3}\log\left(\frac{a+\sqrt{x^2+a^2}}{x}\right).$

168. $\int \frac{dx}{x^3\sqrt{x^2-a^2}}\,dx = \frac{\sqrt{x^2-a^2}}{2a^2x^2} + \frac{1}{2|a|^3}\sec^{-1}\frac{x}{a}.$

169. $\int x^2\sqrt{(x^2\pm a^2)^3}\,dx = \frac{x}{6}\sqrt{(x^2\pm a^2)^5} \mp \frac{a^2x}{24}\sqrt{(x^2\pm a^2)^3} - \frac{a^4x}{16}\sqrt{x^2\pm a^2}$
$\mp\frac{a^6}{16}\log\left(x+\sqrt{x^2\pm a^2}\right).$

170. $\int x^3\sqrt{(x^2\pm a^2)^3}\,dx = \frac{1}{7}\sqrt{(x^2\pm a^2)^7} \mp \frac{a^2}{5}\sqrt{(x^2\pm a^2)^5}.$

171. $\int \frac{\sqrt{x^2\pm a^2}}{x^2}\,dx = -\frac{\sqrt{x^2\pm a^2}}{x} + \log\left(x+\sqrt{x^2\pm a^2}\right).$

172. $\int \frac{\sqrt{x^2+a^2}}{x^3}\,dx = -\frac{\sqrt{x^2+a^2}}{2x^2} - \frac{1}{2a}\log\left(\frac{a+\sqrt{x^2+a^2}}{x}\right).$

173. $\int \frac{\sqrt{x^2-a^2}}{x^3}\,dx = -\frac{\sqrt{x^2-a^2}}{2x^2} + \frac{1}{2\,|a|}\sec^{-1}\frac{x}{a}.$

174. $\int \frac{\sqrt{x^2\pm a^2}}{x^4}\,dx = \mp\frac{\sqrt{(x^2\pm a^2)^3}}{3a^2x^3}.$

175. $\int \frac{x^2\,dx}{\sqrt{(x^2\pm a^2)^3}} = -\frac{x}{\sqrt{x^2\pm a^2}} + \log\left(x+\sqrt{x^2\pm a^2}\right).$

176. $\int \frac{x^3\,dx}{\sqrt{(x^2\pm a^2)^3}} = \sqrt{x^2\pm a^2} \pm \frac{a^2}{\sqrt{x^2\pm a^2}}.$

177. $\int \frac{dx}{x\sqrt{(x^2+a^2)^3}} = \frac{1}{a^2\sqrt{x^2+a^2}} - \frac{1}{a^3}\log\left(\frac{a+\sqrt{x^2+a^2}}{x}\right).$

178. $\int \frac{dx}{x\sqrt{(x^2-a^2)^3}} = -\frac{1}{a^2\sqrt{x^2-a^2}} - \frac{1}{|a^3|}\sec^{-1}\frac{x}{a}.$

179. $\int \frac{dx}{x^2\sqrt{(x^2\pm a^2)^3}} = -\frac{1}{a^4}\left[\frac{\sqrt{x^2\pm a^2}}{x} + \frac{x}{\sqrt{x^2\pm a^2}}\right].$

180. $\int \frac{dx}{x^3\sqrt{(x^2+a^2)^3}} = -\frac{3+a^2}{2a^4\sqrt{x^2+a^2}} + \frac{3}{2a^5}\log\left(\frac{a+\sqrt{x^2+a^2}}{x}\right).$

181. $\int \frac{dx}{x^3\sqrt{(x^2-a^2)^3}} = \frac{1}{2a^2x^2\sqrt{x^2-a^2}} - \frac{3}{2a^4\sqrt{x^2-a^2}} - \frac{3}{2\,|a^5|}\sec^{-1}\frac{x}{a}.$

182. $\int \frac{x^m\,dx}{\sqrt{x^2\pm a^2}} = \frac{1}{m}x^{m-1}\sqrt{x^2\pm a^2} \mp \frac{m-1}{m}a^2\int\frac{x^{m-2}}{\sqrt{x^2\pm a^2}}\,dx.$

183. $$\int \frac{x^{2m}\,dx}{\sqrt{x^2 \pm a^2}} = \frac{(2m)!}{2^{2m}(m!)^2}\left[\sqrt{x^2 \pm a^2}\sum_{r=1}^{m}\frac{r!(r-1)!}{(2r)!}(\mp a^2)^{m-r}(2x)^{2r-1} + (\mp a^2)^m \log\left(x + \sqrt{x^2 \pm a^2}\right)\right].$$

184. $$\int \frac{x^{2m+1}\,dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}\sum_{r=0}^{m}\frac{(2r)!(m!)^2}{(2m+1)!(r!)^2}(\mp 4a^2)^{m-r}x^{2r}.$$

185. $$\int \frac{dx}{x^m\sqrt{x^2 \pm a^2}} = \mp\frac{\sqrt{x^2 \pm a^2}}{(m-1)a^2x^{m-1}} \mp \frac{(m-2)}{(m-1)a^2}\int \frac{dx}{x^{m-2}\sqrt{x^2 \pm a^2}}.$$

186. $$\int \frac{dx}{x^{2m}\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}\sum_{r=0}^{m-1}\frac{(m-1)!m!(2r)!2^{2m-2r-1}}{(r!)^2(2m)!(\mp a^2)^{m-r}x^{2r+1}}.$$

187. $$\int \frac{dx}{x^{2m+1}\sqrt{x^2 + a^2}} = \frac{(2m)!}{(m!)^2}\left[\frac{\sqrt{x^2 + a^2}}{a^2}\sum_{r=1}^{m}(-1)^{m-r+1}\frac{r!(r-1)!}{2(2r)!(4a^2)^{m-r}x^{2r}} + \frac{(-1)^{m+1}}{2^{2m}a^{2m+1}}\log\left(\frac{\sqrt{x^2 + a^2} + a}{x}\right)\right].$$

188. $$\int \frac{dx}{x^{2m+1}\sqrt{x^2 - a^2}} = \frac{(2m)!}{(m!)^2}\left[\frac{\sqrt{x^2 - a^2}}{a^2}\sum_{r=1}^{m}\frac{r!(r-1)!}{2(2r)!(4a^2)^{m-r}x^{2r}} + \frac{1}{2^{2m}\,|a|^{2m+1}}\sec^{-1}\frac{x}{a}\right].$$

189. $$\int \frac{dx}{(x-a)\sqrt{x^2 - a^2}} = -\frac{\sqrt{x^2 - a^2}}{a(x-a)}.$$

190. $$\int \frac{dx}{(x+a)\sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{a(x+a)}.$$

5.4.12 FORMS CONTAINING $\sqrt{a^2 - x^2}$

191. $$\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2\sin^{-1}\frac{x}{|a|}\right).$$

192. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{|a|} = -\cos^{-1}\frac{x}{|a|}.$$

193. $$\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a}\log\left(\frac{a + \sqrt{a^2 - x^2}}{x}\right).$$

194. $$\int \frac{\sqrt{a^2 - x^2}}{x}\,dx = \sqrt{a^2 - x^2} - a\log\left(\frac{a + \sqrt{a^2 - x^2}}{x}\right).$$

195. $$\int \frac{x}{\sqrt{a^2 - x^2}}\,dx = -\sqrt{a^2 - x^2}.$$

196. $$\int x\sqrt{a^2 - x^2}\,dx = -\frac{1}{3}\sqrt{(a^2 - x^2)^3}.$$

197. $$\int \sqrt{(a^2 - x^2)^3}\,dx = \frac{1}{4}\left(x\sqrt{(a^2 - x^2)^3} + \frac{3a^2x}{2}\sqrt{a^2 - x^2} + \frac{3a^4}{2}\sin^{-1}\frac{x}{|a|}\right).$$

198. $$\int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{a^2\sqrt{a^2 - x^2}}.$$

199. $$\int \frac{x}{\sqrt{(a^2 - x^2)^3}}\,dx = \frac{1}{\sqrt{a^2 - x^2}}.$$

200. $$\int x\sqrt{(a^2-x^2)^3}\,dx = -\frac{1}{5}\sqrt{(a^2-x^2)^5}.$$

201. $$\int x^2\sqrt{a^2-x^2}\,dx = -\frac{x}{4}\sqrt{(a^2-x^2)^3} + \frac{a^2}{8}\left(x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{|a|}\right).$$

202. $$\int x^3\sqrt{a^2-x^2}\,dx = \left(-\frac{1}{5}x^2 - \frac{2}{15}a^2\right)\sqrt{(a^2-x^2)^3}.$$

203. $$\int x^2\sqrt{(a^2-x^2)^3}\,dx = -\frac{1}{6}x\sqrt{(a^2-x^2)^5} + \frac{a^2x}{24}\sqrt{(a^2-x^2)^3} + \frac{a^4x}{16}\sqrt{a^2-x^2} + \frac{a^6}{16}\sin^{-1}\frac{x}{|a|}.$$

204. $$\int x^3\sqrt{(a^2-x^2)^3}\,dx = \frac{1}{7}\sqrt{(a^2-x^2)^7} - \frac{a^2}{5}\sqrt{(a^2-x^2)^5}.$$

205. $$\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{|a|}.$$

206. $$\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x}.$$

207. $$\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\frac{\sqrt{a^2-x^2}}{x} - \sin^{-1}\frac{x}{|a|}.$$

208. $$\int \frac{\sqrt{a^2-x^2}}{x^3}\,dx = -\frac{\sqrt{a^2-x^2}}{2x^2} + \frac{1}{2a}\log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right).$$

209. $$\int \frac{\sqrt{a^2-x^2}}{x^4}\,dx = -\frac{\sqrt{(a^2-x^2)^3}}{3a^2x^3}.$$

210. $$\int \frac{x^2\,dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{|a|}.$$

211. $$\int \frac{x^3\,dx}{\sqrt{a^2-x^2}} = -\frac{2}{3}\sqrt{(a^2-x^2)^3} - x^2\sqrt{a^2-x^2}.$$

212. $$\int \frac{x^3\,dx}{\sqrt{(a^2-x^2)^3}} = 2\sqrt{a^2-x^2} + \frac{x^2}{\sqrt{a^2-x^2}} = \frac{a^2}{\sqrt{a^2-x^2}} + \sqrt{a^2-x^2}.$$

213. $$\int \frac{dx}{x^3\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{2a^2x^2} - \frac{1}{2a^3}\log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right).$$

214. $$\int \frac{dx}{x\sqrt{(a^2-x^2)^3}} = \frac{1}{a^2\sqrt{a^2-x^2}} - \frac{1}{a^3}\log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right).$$

215. $$\int \frac{dx}{x^2\sqrt{(a^2-x^2)^3}} = \frac{1}{a^4}\left(-\frac{\sqrt{a^2-x^2}}{x} + \frac{x}{\sqrt{a^2-x^2}}\right).$$

216. $$\int \frac{dx}{x^3\sqrt{(a^2-x^2)^3}} = \frac{3-a^2}{2a^4\sqrt{a^2-x^2}} - \frac{3}{2a^5}\log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right).$$

217. $$\int \frac{x^m}{\sqrt{a^2-x^2}}\,dx = -\frac{x^{m-1}\sqrt{a^2-x^2}}{m} + \frac{(m-1)a^2}{m}\int \frac{x^{m-2}}{\sqrt{a^2-x^2}}\,dx.$$

218. $$\int \frac{x^{2m}}{\sqrt{a^2-x^2}}\,dx = \frac{(2m)!}{(m!)^2}\left[-\sqrt{a^2-x^2}\sum_{r=1}^{m}\frac{r!(r-1)!}{2^{2m-2r+1}(2r)!}a^{2m-2r}x^{2r-1} + \frac{a^{2m}}{2^{2m}}\sin^{-1}\frac{x}{|a|}\right].$$

219. $$\int \frac{x^{2m+1}}{\sqrt{a^2-x^2}}\,dx = -\sqrt{a^2-x^2}\sum_{r=0}^{m}\frac{(2r)!(m!)^2}{(2m+1)!(r!)^2}(4a^2)^{m-r}x^{2r}.$$

220. $$\int \frac{dx}{x^m\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{(m-1)a^2x^{m-1}} + \frac{(m-2)}{(m-1)a^2}\int \frac{dx}{x^{m-2}\sqrt{a^2-x^2}}.$$

221. $$\int \frac{dx}{x^{2m}\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}\sum_{r=0}^{m-1}\frac{(m-1)!m!(2r)!2^{2m-2r-1}}{(r!)^2(2m)!a^{2m-2r}x^{2r+1}}.$$

222. $$\int \frac{dx}{x^{2m+1}\sqrt{a^2-x^2}} = \frac{(2m)!}{(m!)^2}\left[-\frac{\sqrt{a^2-x^2}}{a^2}\sum_{r=1}^{m}\frac{r!(r-1)!}{2(2r)!(4a^2)^{m-r}x^{2r}} + \frac{1}{2^{2m}a^{2m+1}}\log\left(\frac{a-\sqrt{a^2-x^2}}{x}\right)\right].$$

223. $$\int \frac{dx}{(b^2-x^2)\sqrt{a^2-x^2}} = \begin{cases} \dfrac{1}{2b\sqrt{a^2-b^2}}\log\left(\dfrac{(b\sqrt{a^2-x^2}+x\sqrt{a^2-b^2})^2}{b^2-x^2}\right), & a^2>b^2, \\ \text{or} & \\ \dfrac{1}{b\sqrt{b^2-a^2}}\tan^{-1}\dfrac{x\sqrt{b^2-a^2}}{b\sqrt{a^2-x^2}}, & b^2>a^2. \end{cases}$$

224. $$\int \frac{dx}{(b^2+x^2)\sqrt{a^2-x^2}} = \frac{1}{b\sqrt{a^2+b^2}}\tan^{-1}\frac{x\sqrt{a^2+b^2}}{b\sqrt{a^2-x^2}}.$$

225. $$\int \frac{\sqrt{a^2-x^2}}{b^2+x^2}\,dx = \frac{\sqrt{a^2+b^2}}{|b|}\sin^{-1}\frac{x\sqrt{a^2+b^2}}{|a|\sqrt{x^2+b^2}} - \sin^{-1}\frac{x}{|a|}, \quad b^2>a^2.$$

5.4.13 FORMS CONTAINING $\sqrt{a+bx+cx^2}$

$$X = a+bx+cx^2, \quad q = 4ac-b^2, \quad \text{and} \quad k = 4c/q.$$

$$\text{If } q=0, \text{ then } \sqrt{X} = \sqrt{c}\left|x+\tfrac{b}{2c}\right|.$$

226. $$\int \frac{dx}{\sqrt{X}} = \begin{cases} \dfrac{1}{\sqrt{c}}\log\left(\dfrac{2\sqrt{cX}+2cx+b}{\sqrt{q}}\right), & c>0, \\ \text{or} & \\ \dfrac{1}{\sqrt{c}}\sinh^{-1}\dfrac{2cx+b}{\sqrt{q}}, & c>0, \\ \text{or} & \\ -\dfrac{1}{\sqrt{-c}}\sin^{-1}\dfrac{2cx+b}{\sqrt{-q}}, & c<0. \end{cases}$$

227. $$\int \frac{dx}{X\sqrt{X}} = \frac{2(2cx+b)}{q\sqrt{X}}.$$

228. $$\int \frac{dx}{X^2\sqrt{X}} = \frac{2(2cx+b)}{3q\sqrt{X}}\left(\frac{1}{X}+2k\right).$$

229. $$\int \frac{dx}{X^n\sqrt{X}} = \begin{cases} \dfrac{2(2cx+b)\sqrt{X}}{(2n-1)qX^n} + \dfrac{2k(n-1)}{2n-1}\displaystyle\int \frac{dx}{X^{n-1}\sqrt{X}}, \\ \text{or} \\ \dfrac{(2cx+b)(n!)(n-1)!4^n k^{n-1}}{q(2n)!\sqrt{X}}\displaystyle\sum_{r=0}^{n-1}\frac{(2r)!}{(4kX)^r(r!)^2}. \end{cases}$$

230. $\int \sqrt{X}\,dx = \frac{(2cx+b)\sqrt{X}}{4c} + \frac{1}{2k}\int \frac{dx}{\sqrt{X}}$.

231. $\int X\sqrt{X}\,dx = \frac{(2cx+b)\sqrt{X}}{8c}\left(X + \frac{3}{2k}\right) + \frac{3}{8k^2}\int \frac{dx}{\sqrt{X}}$.

232. $\int X^2\sqrt{X}\,dx = \frac{(2cx+b)\sqrt{X}}{12c}\left(X^2 + \frac{5X}{4k} + \frac{15}{8k^2}\right) + \frac{5}{16k^3}\int \frac{dx}{\sqrt{X}}$.

233. $\int X^n\sqrt{X}\,dx =$

$$\begin{cases} \dfrac{(2cx+b)X^n\sqrt{X}}{4(n+1)c} + \dfrac{2n+1}{2(n+1)k}\displaystyle\int X^{n-1}\sqrt{X}\,dx, \\ \text{or} \\ \dfrac{(2n+2)!}{\left[(n+1)!\right]^2(4k)^{n+1}}\left[\dfrac{k(2cx+b)\sqrt{X}}{c}\displaystyle\sum_{r=0}^{n}\dfrac{r!(r+1)!(4kX)^r}{(2r+2)!} + \displaystyle\int\dfrac{dx}{\sqrt{X}}\right]. \end{cases}$$

234. $\int \frac{x\,dx}{\sqrt{X}} = \frac{\sqrt{X}}{c} - \frac{b}{2c}\int \frac{dx}{\sqrt{X}}$.

235. $\int \frac{x\,dx}{X\sqrt{X}} = -\frac{2(bx+2a)}{q\sqrt{X}}$.

236. $\int \frac{x\,dx}{X^n\sqrt{X}} = -\frac{\sqrt{X}}{(2n-1)cX^n} - \frac{b}{2c}\int \frac{dx}{X^n\sqrt{X}}$.

237. $\int \frac{x^2\,dx}{\sqrt{X}} = \left(\frac{x}{2c} - \frac{3b}{4c^2}\right)\sqrt{X} + \frac{3b^2-4ac}{8c^2}\int \frac{dx}{\sqrt{X}}$.

238. $\int \frac{x^2\,dx}{X\sqrt{X}} = \frac{(2b^2-4ac)x+2ab}{cq\sqrt{X}} + \frac{1}{c}\int \frac{dx}{\sqrt{X}}$.

239. $\int \frac{x^2\,dx}{X^n\sqrt{X}} = \frac{(2b^2-4ac)x+2ab}{(2n-1)cqX^{n-1}\sqrt{X}} + \frac{4ac+(2n-3)b^2}{(2n-1)cq}\int \frac{dx}{X^{n-1}\sqrt{X}}$.

240. $\int \frac{x^3\,dx}{\sqrt{X}} = \left(\frac{x^2}{3c} - \frac{5bx}{12c^2} + \frac{5b^2}{8c^3} - \frac{2a}{3c^2}\right)\sqrt{X} + \left(\frac{3ab}{4c^2} - \frac{5b^3}{16c^3}\right)\int \frac{dx}{\sqrt{X}}$.

241. $\int \frac{x^n\,dx}{\sqrt{X}} = \frac{1}{nc}x^{n-1}\sqrt{X} - \frac{(2n-1)b}{2nc}\int \frac{x^{n-1}\,dx}{\sqrt{X}} - \frac{(n-1)a}{nc}\int \frac{x^{n-2}\,dx}{\sqrt{X}}$.

242. $\int x\sqrt{X}\,dx = \frac{X\sqrt{X}}{3c} - \frac{b(2cx+b)}{8c^2}\sqrt{X} - \frac{b}{4ck}\int \frac{dx}{\sqrt{X}}$.

243. $\int xX\sqrt{X}\,dx = \frac{X^2\sqrt{X}}{5c} - \frac{b}{2c}\int X\sqrt{X}\,dx$.

244. $\int xX^n\sqrt{X}\,dx = \frac{X^{n+1}\sqrt{X}}{(2n+3)c} - \frac{b}{2c}\int X^n\sqrt{X}\,dx$.

245. $\int x^2\sqrt{X}\,dx = \left(x - \frac{5b}{6c}\right)\frac{X\sqrt{X}}{4c} + \frac{5b^2-4ac}{16c^2}\int \sqrt{X}\,dx$.

246. $$\int \frac{dx}{x\sqrt{X}} = \begin{cases} \dfrac{1}{\sqrt{-a}}\sin^{-1}\left(\dfrac{bx+2a}{|x|\sqrt{-q}}\right), & a<0, \\ \text{or} & \\ -\dfrac{2\sqrt{X}}{bx}, & a=0, \\ \text{or} & \\ -\dfrac{1}{\sqrt{a}}\log\left(\dfrac{2\sqrt{aX}+bx+2a}{x}\right), & a>0. \end{cases}$$

247. $\displaystyle\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{X}}.$

248. $\displaystyle\int \frac{\sqrt{X}}{x}\,dx = \sqrt{X} + \frac{b}{2}\int \frac{dx}{\sqrt{X}} + a\int \frac{dx}{x\sqrt{X}}.$

249. $\displaystyle\int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} + \frac{b}{2}\int \frac{dx}{x\sqrt{X}} + c\int \frac{dx}{\sqrt{X}}.$

5.4.14 FORMS CONTAINING $\sqrt{2ax - x^2}$

250. $\displaystyle\int \sqrt{2ax - x^2}\,dx = \frac{1}{2}\left[(x-a)\sqrt{2ax - x^2} + a^2\sin^{-1}\frac{x-a}{|a|}\right].$

251. $$\int \frac{dx}{\sqrt{2ax - x^2}} = \begin{cases} \cos^{-1}\left(\dfrac{a-x}{|a|}\right), \\ \quad\text{or} \\ \sin^{-1}\left(\dfrac{x-a}{|a|}\right). \end{cases}$$

252. $\displaystyle\int x^n\sqrt{2ax - x^2}\,dx =$
$$\begin{cases} -\dfrac{x^{n-1}\sqrt{(2ax-x^2)^3}}{n+2} + \dfrac{(2n+1)a}{n+2}\displaystyle\int x^{n-1}\sqrt{2ax - x^2}\,dx, \\ \text{or} \\ \sqrt{2ax - x^2}\left[\dfrac{x^{n+1}}{n+2} - \displaystyle\sum_{r=0}^{n}\dfrac{(2n+1)!(r!)^2a^{n-r+1}}{2^{n-r}(2r+1)!(n+2)!n!}x^r\right] + \dfrac{(2n+1)!a^{n+2}}{2^n n!(n+2)!}\sin^{-1}\left(\dfrac{x-a}{|a|}\right). \end{cases}$$

253. $\displaystyle\int \frac{\sqrt{2ax - x^2}}{x^n}\,dx = \frac{\sqrt{(2ax-x^2)^3}}{(3-2n)ax^n} + \frac{n-3}{(2n-3)a}\int \frac{\sqrt{2ax - x^2}}{x^{n-1}}\,dx.$

254. $\displaystyle\int \frac{x^n\,dx}{\sqrt{2ax - x^2}} =$
$$\begin{cases} -\dfrac{x^{n-1}\sqrt{2ax - x^2}}{n} + \dfrac{a(2n-1)}{n}\displaystyle\int \dfrac{x^{n-1}}{\sqrt{2ax - x^2}}\,dx, \\ \text{or} \\ -\sqrt{2ax - x^2}\displaystyle\sum_{r=1}^{n}\dfrac{(2n)!r!(r-1)!a^{n-r}}{2^{n-r}(2r)!(n!)^2}x^{r-1} + \dfrac{(2n)!a^n}{2^n(n!)^2}\sin^{-1}\left(\dfrac{x-a}{|a|}\right). \end{cases}$$

255. $$\int \frac{dx}{x^n\sqrt{2ax - x^2}} = \begin{cases} \dfrac{\sqrt{2ax - x^2}}{a(1-2n)x^n} + \dfrac{n-1}{(2n-1)a}\displaystyle\int \dfrac{dx}{x^{n-1}\sqrt{2ax - x^2}}, \\ \text{or} \\ -\sqrt{2ax - x^2}\displaystyle\sum_{r=0}^{n-1}\dfrac{2^{n-r}(n-1)!n!(2r)!}{(2n)!(r!)^2a^{n-r}x^{r+1}}. \end{cases}$$

256. $\displaystyle\int \frac{dx}{\sqrt{(2ax-x^2)^3}} = \frac{x-a}{a^2\sqrt{2ax - x^2}}.$

257. $\displaystyle\int \frac{x\,dx}{\sqrt{(2ax-x^2)^3}} = \frac{x}{a\sqrt{2ax - x^2}}.$

5.4.15 MISCELLANEOUS ALGEBRAIC FORMS

258. $\displaystyle\int \frac{dx}{\sqrt{2ax + x^2}} = \log\left(x + a + \sqrt{2ax + x^2}\right).$

259. $$\int \sqrt{ax^2+c}\,dx = \begin{cases} \dfrac{x}{2}\sqrt{ax^2+c} + \dfrac{c}{2\sqrt{-a}}\sin^{-1}\left(x\sqrt{-\dfrac{a}{c}}\right), & a<0, \\ \text{or} & \\ \dfrac{x}{2}\sqrt{ax^2+c} + \dfrac{c}{2\sqrt{a}}\log\left(x\sqrt{a}+\sqrt{ax^2+c}\right), & a>0. \end{cases}$$

260. $$\int \sqrt{\frac{1+x}{1-x}}\,dx = \sin^{-1}x - \sqrt{1-x^2}.$$

261. $$\int \frac{dx}{x\sqrt{ax^n+c}} = \begin{cases} \dfrac{1}{n\sqrt{c}}\log\dfrac{\sqrt{ax^n+c}-\sqrt{c}}{\sqrt{ax^n+c}+\sqrt{c}}, & \\ \text{or} & \\ \dfrac{2}{n\sqrt{c}}\log\dfrac{\sqrt{ax^n+c}-\sqrt{c}}{\sqrt{x^n}}, & c>0, \\ \text{or} & \\ \dfrac{2}{n\sqrt{-c}}\sec^{-1}\sqrt{-\dfrac{ax^n}{c}}, & c<0. \end{cases}$$

262. $$\int \frac{dx}{\sqrt{ax^2+c}} = \begin{cases} \dfrac{1}{\sqrt{-a}}\sin^{-1}\left(x\sqrt{-\dfrac{a}{c}}\right), & a<0, \\ \text{or} & \\ \dfrac{1}{\sqrt{a}}\log\left(x\sqrt{a}+\sqrt{ax^2+c}\right), & a>0. \end{cases}$$

263. $$\int (ax^2+c)^{m+1/2}\,dx =$$
$$\begin{cases} \dfrac{x(ax^2+c)^{m+1/2}}{2(m+1)} + \dfrac{(2m+1)c}{2(m+1)}\displaystyle\int (ax^2+c)^{m-1/2}\,dx, \\ \text{or} \\ x\sqrt{ax^2+c}\displaystyle\sum_{r=0}^{m}\dfrac{(2m+1)!(r!)^2c^{m-r}}{2^{2m-2r+1}m!(m+1)!(2r+1)!}(ax^2+c)^r \\ \qquad + \dfrac{(2m+1)!c^{m+1}}{2^{2m+1}m!(m+1)!}\displaystyle\int\dfrac{dx}{\sqrt{ax^2+c}}. \end{cases}$$

264. $$\int x(ax^2+c)^{m+1/2}\,dx = \frac{(ax^2+c)^{m+3/2}}{(2m+3)a}.$$

265. $$\int \frac{(ax^2+c)^{m+1/2}}{x}\,dx =$$
$$\begin{cases} \dfrac{(ax^2+c)^{m+1/2}}{2m+1} + c\displaystyle\int\dfrac{(ax^2+c)^{m-1/2}}{x}\,dx, \\ \text{or} \\ \sqrt{ax^2+c}\displaystyle\sum_{r=0}^{m}\dfrac{c^{m-r}(ax^2+c)^r}{2r+1} + c^{m+1}\displaystyle\int\dfrac{dx}{x\sqrt{ax^2+c}}. \end{cases}$$

266. $$\int \frac{dx}{(ax^2+c)^{m+1/2}} =$$
$$\begin{cases} \dfrac{x}{(2m-1)c(ax^2+c)^{m-1/2}} + \dfrac{2m-2}{(2m-1)c}\displaystyle\int\dfrac{dx}{(ax^2+c)^{m-1/2}}, \\ \text{or} \\ \dfrac{x}{\sqrt{ax^2+c}}\displaystyle\sum_{r=0}^{m-1}\dfrac{2^{2m-2r-1}(m-1)!m!(2r)!}{(2m)!(r!)^2c^{m-r}(ax^2+c)^r}. \end{cases}$$

267. $$\int \frac{dx}{x^m\sqrt{ax^2+c}} = -\frac{\sqrt{ax^2+c}}{(m-1)cx^{m-1}} - \frac{(m-2)a}{(m-1)c}\int\frac{dx}{x^{m-2}\sqrt{ax^2+c}}, \quad m\neq 1.$$

268. $\displaystyle\int \frac{1+x^2}{(1-x^2)\sqrt{1+x^4}}\,dx = \frac{1}{\sqrt{2}}\log\left(\frac{x\sqrt{2}+\sqrt{1+x^4}}{1-x^2}\right).$

269. $\displaystyle\int \frac{1-x^2}{(1+x^2)\sqrt{1+x^4}}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{\sqrt{1+x^4}}.$

270. $\displaystyle\int \frac{dx}{x\sqrt{x^n+a^2}} = -\frac{2}{na}\log\left(\frac{a+\sqrt{x^n+a^2}}{\sqrt{x^n}}\right).$

271. $\displaystyle\int \frac{dx}{x\sqrt{x^n-a^2}} = -\frac{2}{na}\sin^{-1}\frac{a}{\sqrt{x^n}}.$

272. $\displaystyle\int \sqrt{\frac{x}{a^3-x^3}}\,dx = \frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{3/2}.$

5.4.16 FORMS INVOLVING TRIGONOMETRIC FUNCTIONS

273. $\displaystyle\int \sin ax\,dx = -\frac{1}{a}\cos ax.$

274. $\displaystyle\int \cos ax\,dx = \frac{1}{a}\sin ax.$

275. $\displaystyle\int \tan ax\,dx = -\frac{1}{a}\log\cos ax = \frac{1}{a}\log\sec ax.$

276. $\displaystyle\int \cot ax\,dx = \frac{1}{a}\log\sin ax = -\frac{1}{a}\log\csc ax.$

277. $\displaystyle\int \sec ax\,dx = \frac{1}{a}\log(\sec ax+\tan ax) = \frac{1}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right).$

278. $\displaystyle\int \csc ax\,dx = \frac{1}{a}\log(\csc ax-\cot ax) = \frac{1}{a}\log\tan\frac{ax}{2}.$

279. $\displaystyle\int \sin^2 ax\,dx = \frac{x}{2}-\frac{1}{2a}\cos ax\sin ax = \frac{x}{2}-\frac{1}{4a}\sin 2ax.$

280. $\displaystyle\int \sin^3 ax\,dx = -\frac{1}{3a}(\cos ax)(\sin^2 ax+2).$

281. $\displaystyle\int \sin^4 ax\,dx = \frac{3x}{8}-\frac{\sin 2ax}{4a}+\frac{\sin 4ax}{32a}.$

282. $\displaystyle\int \sin^n ax\,dx = -\frac{\sin^{n-1}ax\cos ax}{na}+\frac{n-1}{n}\int \sin^{n-2}ax\,dx.$

283. $\displaystyle\int \sin^{2m} ax\,dx = -\frac{\cos ax}{a}\sum_{r=0}^{m-1}\frac{(2m)!(r!)^2}{2^{2m-2r}(2r+1)!(m!)^2}\sin^{2r+1}ax+\frac{(2m)!}{2^{2m}(m!)^2}x.$

284. $\displaystyle\int \sin^{2m+1} ax\,dx = -\frac{\cos ax}{a}\sum_{r=0}^{m-1}\frac{2^{2m-2r}(m!)^2(2r)!}{(2m+1)!(r!)^2}\sin^{2r}ax.$

285. $\displaystyle\int \cos^2 ax\,dx = \frac{1}{2}x+\frac{1}{2a}\sin ax\cos ax = \frac{1}{2}x+\frac{1}{4a}\sin 2ax.$

286. $\displaystyle\int \cos^3 ax\,dx = \frac{1}{3a}\sin ax(\cos^2 ax+2).$

287. $\displaystyle\int \cos^4 ax\,dx = \frac{3}{8}x+\frac{\sin 2ax}{4a}+\frac{\sin 4ax}{32a}.$

288. $\displaystyle\int \cos^n ax\,dx = \frac{1}{na}\cos^{n-1}ax\sin ax+\frac{n-1}{n}\int \cos^{n-2}ax\,dx.$

289. $$\int \cos^{2m} ax\,dx = \frac{\sin ax}{a}\sum_{r=0}^{m-1}\frac{(2m)!(r!)^2}{2^{2m-2r}(2r+1)!(m!)^2}\cos^{2r+1} ax + \frac{(2m)!}{2^{2m}(m!)^2}x.$$

290. $$\int \cos^{2m+1} ax\,dx = \frac{\sin ax}{a}\sum_{r=0}^{m}\frac{2^{2m-2r}(m!)^2(2r)!}{(2m+1)!(r!)^2}\cos^{2r} ax.$$

291. $$\int \frac{dx}{\sin^2 ax} = \int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax.$$

292. $$\int \frac{dx}{\sin^m ax} = \int \operatorname{cosec}^m ax\,dx = -\frac{1}{a(m-1)}\frac{\cos ax}{\sin^{m-1} ax} + \frac{m-2}{m-1}\int \frac{dx}{\sin^{m-2} ax}.$$

293. $$\int \frac{dx}{\sin^{2m} ax} = \int \operatorname{cosec}^{2m} ax\,dx = -\frac{1}{a}\cos ax\sum_{r=0}^{m-1}\frac{2^{2m-2r-1}(m-1)!m!(2r)!}{(2m)!(r!)^2\sin^{2r+1} ax}.$$

294. $$\int \frac{dx}{\sin^{2m+1} ax} = \int \operatorname{cosec}^{2m+1} ax\,dx =$$
$$-\frac{1}{a}\cos ax\sum_{r=0}^{m-1}\frac{(2m)!(r!)^2}{2^{2m-2r}(2r+1)!(m!)^2\sin^{2r+2} ax} + \frac{1}{a}\frac{(2m)!}{2^{2m}(m!)^2}\log\tan\frac{ax}{2}.$$

295. $$\int \frac{dx}{\cos^2 ax} = \int \sec^2 ax\,dx = \frac{1}{a}\tan ax.$$

296. $$\int \frac{dx}{\cos^m ax} = \int \sec^m ax\,dx = \frac{1}{a(m-1)}\frac{\sin ax}{\cos^{m-1} ax} + \frac{m-2}{m-1}\int \frac{dx}{\cos^{m-2} ax}.$$

297. $$\int \frac{dx}{\cos^{2m} ax} = \int \sec^{2m} ax\,dx = \frac{1}{a}\sin ax\sum_{r=0}^{m-1}\frac{2^{2m-2r-1}(m-1)!m!(2r)!}{(2m)!(r!)^2\cos^{2r+1} ax}.$$

298. $$\int \frac{dx}{\cos^{2m+1} ax} = \int \sec^{2m+1} ax\,dx = \frac{1}{a}\frac{(2m)!}{2^{2m}(m!)^2}\log(\sec ax + \tan ax)$$
$$+\frac{1}{a}\sin ax\sum_{r=0}^{m-1}\frac{(2m)!(r!)^2}{2^{2m-2r}(m!)^2(2r+1)!\cos^{2r+2} ax}.$$

299. $$\int (\sin mx)(\sin nx)\,dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}, \quad m^2 \neq n^2.$$

300. $$\int (\cos mx)(\cos nx)\,dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}, \quad m^2 \neq n^2.$$

301. $$\int (\sin ax)(\cos ax)\,dx = \frac{1}{2a}\sin^2 ax.$$

302. $$\int (\sin mx)(\cos nx)\,dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)}, \quad m^2 \neq n^2.$$

303. $$\int (\sin^2 ax)(\cos^2 ax)\,dx = -\frac{1}{32a}\sin 4ax + \frac{x}{8}.$$

304. $$\int (\sin ax)(\cos^m ax)\,dx = -\frac{\cos^{m+1} ax}{(m+1)a}.$$

305. $$\int (\sin^m ax)(\cos ax)\,dx = \frac{\sin^{m+1} ax}{(m+1)a}.$$

306. $$\int (\cos^m ax)(\sin^n ax)\,dx =$$
$$\begin{cases} \dfrac{\cos^{m-1} ax\sin^{n+1} ax}{(m+n)a} + \dfrac{m-1}{m+n}\displaystyle\int (\cos^{m-2} ax)(\sin^n ax)\,dx, \\ \text{or} \\ -\dfrac{\cos^{m+1} ax\sin^{n-1} ax}{(m+n)a} + \dfrac{n-1}{m+n}\displaystyle\int (\cos^m ax)(\sin^{n-2} ax)\,dx. \end{cases}$$

307. $$\int \frac{\cos^m ax}{\sin^n ax}\,dx = \begin{cases} -\dfrac{\cos^{m+1} ax}{a(n-1)\sin^{n-1} ax} - \dfrac{m-n+2}{n-1}\displaystyle\int \frac{\cos^m ax}{\sin^{n-2} ax}\,dx, \\ \text{or} \\ \dfrac{\cos^{m-1} ax}{a(m-n)\sin^{n-1} ax} + \dfrac{m-1}{m-n}\displaystyle\int \frac{\cos^{m-2} ax}{\sin^n ax}\,dx. \end{cases}$$

308. $$\int \frac{\sin^m ax}{\cos^n ax}\,dx = \begin{cases} \dfrac{\sin^{m+1} ax}{a(n-1)\cos^{n-1} ax} - \dfrac{m-n+2}{n-1}\displaystyle\int \frac{\sin^m ax}{\cos^{n-2} ax}\,dx, \\ \text{or} \\ -\dfrac{\sin^{m-1} ax}{a(m-n)\cos^{n-1} ax} + \dfrac{m-1}{m-n}\displaystyle\int \frac{\sin^{m-2} ax}{\cos^n ax}\,dx. \end{cases}$$

309. $$\int \frac{\sin ax}{\cos^2 ax}\,dx = \frac{1}{a\cos ax} = \frac{\sec ax}{a}.$$

310. $$\int \frac{\sin^2 ax}{\cos ax}\,dx = -\frac{1}{a}\sin ax + \frac{1}{a}\log\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right).$$

311. $$\int \frac{\cos ax}{\sin^2 ax}\,dx = -\frac{\csc ax}{a} = -\frac{1}{a\sin ax}.$$

312. $$\int \frac{dx}{(\sin ax)(\cos ax)} = \frac{1}{a}\log\tan ax.$$

313. $$\int \frac{dx}{(\sin ax)(\cos^2 ax)} = \frac{1}{a}\left(\sec ax + \log\tan\frac{ax}{2}\right).$$

314. $$\int \frac{dx}{(\sin ax)(\cos^n ax)} = \frac{1}{a(n-1)\cos^{n-1} ax} + \int \frac{dx}{(\sin ax)(\cos^{n-2} ax)}.$$

315. $$\int \frac{dx}{(\sin^2 ax)(\cos ax)} = -\frac{1}{a}\csc ax + \frac{1}{a}\log\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right).$$

316. $$\int \frac{dx}{(\sin^2 ax)(\cos^2 ax)} = -\frac{2}{a}\cot 2ax.$$

317. $$\int \frac{dx}{\sin^m ax\cos^n ax} =$$
$$\begin{cases} -\dfrac{1}{a(m-1)\sin^{m-1} ax\cos^{n-1} ax} + \dfrac{m+n-2}{m-1}\displaystyle\int \frac{dx}{\sin^{m-2} ax\cos^n ax}, \\ \text{or} \\ \dfrac{1}{a(n-1)\sin^{m-1} ax\cos^{n-1} ax} + \dfrac{m+n-2}{n-1}\displaystyle\int \frac{dx}{\sin^m ax\cos^{n-2} ax}. \end{cases}$$

318. $$\int \sin(a+bx)\,dx = -\frac{1}{b}\cos(a+bx).$$

319. $$\int \cos(a+bx)\,dx = \frac{1}{b}\sin(a+bx).$$

320. $$\int \frac{dx}{1 \pm \sin ax} = \mp\frac{1}{a}\tan\left(\frac{\pi}{4} \mp \frac{ax}{2}\right).$$

321. $$\int \frac{dx}{1+\cos ax} = \frac{1}{a}\tan\frac{ax}{2}.$$

322. $$\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\cot\frac{ax}{2}.$$

323. $$\int \frac{dx}{a+b\sin x} = \begin{cases} \dfrac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\dfrac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}}\right), \\ \text{or} \\ \dfrac{1}{\sqrt{b^2-a^2}}\log\left(\dfrac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}}\right). \end{cases}$$

324. $$\int \frac{dx}{a+b\cos x} = \begin{cases} \dfrac{2}{\sqrt{a^2-b^2}} \tan^{-1} \dfrac{\sqrt{a^2-b^2}\tan\frac{x}{2}}{a+b}, \\ \quad\text{or} \\ \dfrac{1}{\sqrt{b^2-a^2}} \log\left(\dfrac{\sqrt{b^2-a^2}\tan\frac{x}{2}+a+b}{\sqrt{b^2-a^2}\tan\frac{x}{2}-a-b}\right). \end{cases}$$

325. $$\int \frac{dx}{a+b\sin x+c\cos x} =$$
$$\begin{cases} \dfrac{1}{\sqrt{b^2+c^2-a^2}} \log\left(\dfrac{b-\sqrt{b^2+c^2-a^2}+(a-c)\tan\frac{x}{2}}{b+\sqrt{b^2+c^2-a^2}+(a-c)\tan\frac{x}{2}}\right), & a \neq c,\ a^2 < b^2+c^2, \\ \quad\text{or} \\ \dfrac{2}{\sqrt{a^2-b^2-c^2}} \tan^{-1} \dfrac{b+(a-c)\tan\frac{x}{2}}{\sqrt{a^2-b^2-c^2}}, & a^2 > b^2+c^2, \\ \quad\text{or} \\ \dfrac{1}{a}\left[\dfrac{a-(b+c)\sin x-(b-c)\sin x}{a-(b+c)\sin x+(b-c)\sin x}\right]. & a^2 = b^2+c^2. \end{cases}$$

326. $$\int \frac{\sin^2 x}{a+b\cos^2 x}\,dx = \frac{1}{b}\sqrt{\frac{a+b}{a}}\tan^{-1}\left(\sqrt{\frac{a}{a+b}}\tan x\right) - \frac{x}{b}, \quad ab>0,\ |a|>|b|.$$

327. $$\int \frac{dx}{a^2\cos^2 x+b^2\sin^2 x} = \frac{1}{ab}\tan^{-1}\left(\frac{b\tan x}{a}\right).$$

328. $$\int \frac{\cos^2 cx}{a^2+b^2\sin^2 cx}\,dx = \frac{\sqrt{a^2+b^2}}{ab^2c}\tan^{-1}\frac{\sqrt{a^2+b^2}\tan cx}{a} - \frac{x}{b^2}.$$

329. $$\int \frac{\sin cx\cos cx}{a\cos^2 cx+b\sin^2 cx}\,dx = \frac{1}{2c(b-a)}\log\left(a\cos^2 cx+b\sin^2 cx\right), \quad a\neq b.$$

330. $$\int \frac{\cos cx}{a\cos cx+b\sin cx}\,dx = \int \frac{dx}{a+b\tan cx} = \frac{1}{c(a^2+b^2)}\left[acx+b\log\left(a\cos cx+b\sin cx\right)\right].$$

331. $$\int \frac{\sin cx}{a\cos cx+b\sin cx}\,dx = \int \frac{dx}{b+a\cot cx} = \frac{1}{c(a^2+b^2)}\left[bcx-a\log\left(a\cos cx+b\sin cx\right)\right].$$

332. $$\int \frac{dx}{a\cos^2 x+2b\cos x\sin x+c\sin^2 x} = \begin{cases} \dfrac{1}{2\sqrt{b^2-ac}}\log\left(\dfrac{c\tan x+b-\sqrt{b^2-ac}}{c\tan x+b+\sqrt{b^2-ac}}\right), & b^2>ac, \\ \quad\text{or} \\ \dfrac{1}{\sqrt{ac-b^2}}\tan^{-1}\left(\dfrac{c\tan x+b}{\sqrt{ac-b^2}}\right), & b^2<ac, \\ \quad\text{or} \\ -\dfrac{1}{c\tan x+b}, & b^2=ac. \end{cases}$$

333. $$\int \frac{\sin ax}{1\pm\sin ax}\,dx = \pm x + \frac{1}{a}\tan\left(\frac{\pi}{4}\mp\frac{ax}{2}\right).$$

334. $$\int \frac{dx}{(\sin ax)(1\pm\sin ax)} = \frac{1}{a}\tan\left(\frac{\pi}{4}\mp\frac{ax}{2}\right) + \frac{1}{a}\log\tan\frac{ax}{2}.$$

335. $$\int \frac{dx}{(1+\sin ax)^2} = -\frac{1}{2a}\tan\left(\frac{\pi}{4}-\frac{ax}{2}\right) - \frac{1}{6a}\tan^3\left(\frac{\pi}{4}-\frac{ax}{2}\right).$$

336. $\displaystyle\int \frac{dx}{(1-\sin ax)^2} = \frac{1}{2a}\cot\left(\frac{\pi}{4}-\frac{ax}{2}\right) + \frac{1}{6a}\cot^3\left(\frac{\pi}{4}-\frac{ax}{2}\right).$

337. $\displaystyle\int \frac{\sin ax}{(1+\sin ax)^2}\,dx = -\frac{1}{2a}\tan\left(\frac{\pi}{4}-\frac{ax}{2}\right) + \frac{1}{6a}\tan^3\left(\frac{\pi}{4}-\frac{ax}{2}\right).$

338. $\displaystyle\int \frac{\sin ax}{(1-\sin ax)^2}\,dx = -\frac{1}{2a}\cot\left(\frac{\pi}{4}-\frac{ax}{2}\right) + \frac{1}{6a}\cot^3\left(\frac{\pi}{4}-\frac{ax}{2}\right).$

339. $\displaystyle\int \frac{\sin x}{a+b\sin x}\,dx = \frac{x}{b} - \frac{a}{b}\int \frac{dx}{a+b\sin x}.$

340. $\displaystyle\int \frac{dx}{(\sin x)(a+b\sin x)} = \frac{1}{a}\log\tan\frac{x}{2} - \frac{b}{a}\int \frac{dx}{a+b\sin x}.$

341. $\displaystyle\int \frac{dx}{(a+b\sin x)^2} = \begin{cases} \dfrac{b\cos x}{(a^2-b^2)(a+b\sin x)} + \dfrac{a}{a^2-b^2}\displaystyle\int \frac{dx}{a+b\sin x}, \\ \text{or} \\ \dfrac{a\cos x}{(b^2-a^2)(a+b\sin x)} + \dfrac{b}{b^2-a^2}\displaystyle\int \frac{dx}{a+b\sin x}. \end{cases}$

342. $\displaystyle\int \frac{dx}{a^2+b^2\sin^2 cx} = \frac{1}{ac\sqrt{a^2+b^2}}\tan^{-1}\left(\frac{\sqrt{a^2+b^2}\tan cx}{a}\right).$

343. $\displaystyle\int \frac{dx}{a^2-b^2\sin^2 cx} = \begin{cases} \dfrac{1}{ac\sqrt{a^2-b^2}}\tan^{-1}\left(\dfrac{\sqrt{a^2-b^2}\tan cx}{a}\right), & a^2>b^2, \\ \text{or} \\ \dfrac{1}{2ac\sqrt{b^2-a^2}}\log\left(\dfrac{\sqrt{b^2-a^2}\tan cx+a}{\sqrt{b^2-a^2}\tan cx-a}\right), & a^2<b^2. \end{cases}$

344. $\displaystyle\int \frac{\cos ax}{1+\cos ax}\,dx = x - \frac{1}{a}\tan\frac{ax}{2}.$

345. $\displaystyle\int \frac{\cos ax}{1-\cos ax}\,dx = -x - \frac{1}{a}\cot\frac{ax}{2}.$

346. $\displaystyle\int \frac{dx}{(\cos ax)(1+\cos ax)} = \frac{1}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right) - \frac{1}{a}\tan\frac{ax}{2}.$

347. $\displaystyle\int \frac{dx}{(\cos ax)(1-\cos ax)} = \frac{1}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right) - \frac{1}{a}\cot\frac{ax}{2}.$

348. $\displaystyle\int \frac{dx}{(1+\cos ax)^2} = \frac{1}{2a}\tan\frac{ax}{2} + \frac{1}{6a}\tan^3\frac{ax}{2}.$

349. $\displaystyle\int \frac{dx}{(1-\cos ax)^2} = -\frac{1}{2a}\cot\frac{ax}{2} - \frac{1}{6a}\cot^3\frac{ax}{2}.$

350. $\displaystyle\int \frac{\cos ax}{(1+\cos ax)^2}\,dx = \frac{1}{2a}\tan\frac{ax}{2} - \frac{1}{6a}\tan^3\frac{ax}{2}.$

351. $\displaystyle\int \frac{\cos ax}{(1-\cos ax)^2}\,dx = \frac{1}{2a}\cot\frac{ax}{2} - \frac{1}{6a}\cot^3\frac{ax}{2}.$

352. $\displaystyle\int \frac{\cos x}{a+b\cos x}\,dx = \frac{x}{b} - \frac{a}{b}\int \frac{dx}{a+b\cos x}.$

353. $\displaystyle\int \frac{dx}{(\cos x)(a+b\cos x)} = \frac{1}{a}\log\tan\left(\frac{x}{2}+\frac{\pi}{4}\right) - \frac{b}{a}\int \frac{dx}{a+b\cos x}.$

354. $\displaystyle\int \frac{dx}{(a+b\cos x)^2} = \frac{b\sin x}{(b^2-a^2)(a+b\cos x)} - \frac{a}{b^2-a^2}\int \frac{dx}{a+b\cos x}.$

355. $\displaystyle\int \frac{\cos x}{(a+b\cos x)^2}\,dx = \frac{a\sin x}{(a^2-b^2)(a+b\cos x)} - \frac{b}{a^2-b^2}\int \frac{dx}{a+b\cos x}.$

356. $\displaystyle\int \frac{dx}{a^2+b^2-2ab\cos cx} = \frac{2}{c(a^2-b^2)}\tan^{-1}\left(\frac{a+b}{a-b}\tan\frac{cx}{2}\right).$

357. $\displaystyle\int \frac{dx}{a^2+b^2\cos^2 cx} = \frac{1}{ac\sqrt{a^2+b^2}}\tan^{-1}\frac{a\tan cx}{\sqrt{a^2+b^2}}.$

358. $\displaystyle\int \frac{dx}{a^2-b^2\cos^2 cx} = \begin{cases} \dfrac{1}{ac\sqrt{a^2-b^2}}\tan^{-1}\left(\dfrac{a\tan cx}{\sqrt{a^2-b^2}}\right), & a^2>b^2, \\ \text{or} & \\ \dfrac{1}{2ac\sqrt{b^2-a^2}}\log\left(\dfrac{a\tan cx-\sqrt{b^2-a^2}}{a\tan cx+\sqrt{b^2-a^2}}\right), & b^2>a^2. \end{cases}$

359. $\displaystyle\int \frac{\sin ax}{1\pm\cos ax}\,dx = \mp\frac{1}{a}\log(1\pm\cos ax).$

360. $\displaystyle\int \frac{\cos ax}{1\pm\sin ax}\,dx = \pm\frac{1}{a}\log(1\pm\sin ax).$

361. $\displaystyle\int \frac{dx}{(\sin ax)(1\pm\cos ax)} = \pm\frac{1}{2a(1\pm\cos ax)}+\frac{1}{2a}\log\tan\frac{ax}{2}.$

362. $\displaystyle\int \frac{dx}{(\cos ax)(1\pm\sin ax)} = \mp\frac{1}{2a(1\pm\sin ax)}+\frac{1}{2a}\log\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right).$

363. $\displaystyle\int \frac{\sin ax}{(\cos ax)(1\pm\cos ax)}\,dx = \frac{1}{a}\log(\sec ax\pm 1).$

364. $\displaystyle\int \frac{\cos ax}{(\sin ax)(1\pm\sin ax)}\,dx = -\frac{1}{a}\log(\csc ax\pm 1).$

365. $\displaystyle\int \frac{\sin ax}{(\cos ax)(1\pm\sin ax)}\,dx = \frac{1}{2a(1\pm\sin ax)}\pm\frac{1}{2a}\log\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right).$

366. $\displaystyle\int \frac{\cos ax}{(\sin ax)(1\pm\cos ax)}\,dx = -\frac{1}{2a(1\pm\cos ax)}\pm\frac{1}{2a}\log\tan\frac{ax}{2}.$

367. $\displaystyle\int \frac{dx}{\sin ax\pm\cos ax} = \frac{1}{a\sqrt{2}}\log\tan\left(\frac{ax}{2}\pm\frac{\pi}{8}\right).$

368. $\displaystyle\int \frac{dx}{(\sin ax\pm\cos ax)^2} = \frac{1}{2a}\tan\left(ax\mp\frac{\pi}{4}\right).$

369. $\displaystyle\int \frac{dx}{1+\cos ax\pm\sin ax} = \pm\frac{1}{a}\log\left(1\pm\tan\frac{ax}{2}\right).$

370. $\displaystyle\int \frac{dx}{a^2\cos^2 cx-b^2\sin^2 cx} = \frac{1}{2abc}\log\left(\frac{b\tan cx+a}{b\tan cx-a}\right).$

371. $\displaystyle\int x\sin ax\,dx = \frac{1}{a^2}\sin ax-\frac{x}{a}\cos ax.$

372. $\displaystyle\int x^2\sin ax\,dx = \frac{2x}{a^2}\sin ax+\frac{2-a^2x^2}{a^3}\cos ax.$

373. $\displaystyle\int x^3\sin ax\,dx = \frac{3a^2x^2-6}{a^4}\sin ax+\frac{6x-a^2x^3}{a^3}\cos ax.$

374. $\displaystyle\int x^m\sin ax\,dx =$

$$\begin{cases} -\dfrac{1}{a}x^m\cos ax+\dfrac{m}{a}\displaystyle\int x^{m-1}\cos ax\,dx, \\ \text{or} \\ \cos ax\displaystyle\sum_{r=0}^{\lfloor\frac{m}{2}\rfloor}\frac{(-1)^{r+1}m!}{(m-2r)!}\frac{x^{m-2r}}{a^{2r+1}}+\sin ax\sum_{r=0}^{\lfloor\frac{m-1}{2}\rfloor}\frac{(-1)^r m!}{(m-2r-1)!}\frac{x^{m-2r-1}}{a^{2r+2}}. \end{cases}$$

375. $\int x\cos ax\,dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax.$

376. $\int x^2\cos ax\,dx = \frac{2x}{a^2}\cos ax + \frac{a^2x^2-2}{a^3}\sin ax.$

377. $\int x^3\cos ax\,dx = \frac{3a^2x^2-6}{a^4}\cos ax + \frac{a^2x^3-6x}{a^3}\sin ax.$

378. $\int x^m\cos ax\,dx =$

$$\begin{cases} \frac{x^m}{a}\sin ax - \frac{m}{a}\int x^{m-1}\sin ax\,dx, \\ \text{or} \\ \sin ax\sum_{r=0}^{\lfloor \frac{m}{2}\rfloor}\frac{(-1)^r m!}{(m-2r)!}\frac{x^{m-2r}}{a^{2r+1}} + \cos ax\sum_{r=0}^{\lfloor \frac{m-1}{2}\rfloor}\frac{(-1)^r m!}{(m-2r-1)!}\frac{x^{m-2r-1}}{a^{2r+2}}. \end{cases}$$

379. $\int \frac{\sin ax}{x}\,dx = \sum_{n=0}^{\infty}(-1)^n\frac{(ax)^{2n+1}}{(2n+1)(2n+1)!}.$

380. $\int \frac{\cos ax}{x}\,dx = \sum_{n=0}^{\infty}(-1)^n\frac{(ax)^{2n}}{(2n)(2n)!}.$

381. $\int x\sin^2 ax\,dx = \frac{x^2}{4} - \frac{x}{4a}\sin 2ax - \frac{1}{8a^2}\cos 2ax.$

382. $\int x^2\sin^2 ax\,dx = \frac{x^3}{6} - \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin 2ax - \frac{x}{4a^2}\cos 2ax.$

383. $\int x\sin^3 ax\,dx = \frac{x}{12a}\cos 3ax - \frac{1}{36a^2}\sin 3ax - \frac{3x}{4a}\cos ax + \frac{3}{4a^2}\sin ax.$

384. $\int x\cos^2 ax\,dx = \frac{x^2}{4} + \frac{x}{4a}\sin 2ax + \frac{1}{8a^2}\cos 2ax.$

385. $\int x^2\cos^2 ax\,dx = \frac{x^3}{6} + \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin 2ax + \frac{x}{4a^2}\cos 2ax.$

386. $\int x\cos^3 ax\,dx = \frac{x}{12a}\sin 3ax + \frac{1}{36a^2}\cos 3ax + \frac{3x}{4a}\sin ax + \frac{3}{4a^2}\cos ax.$

387. $\int \frac{\sin ax}{x^m}\,dx = \frac{\sin ax}{(1-m)x^{m-1}} + \frac{a}{m-1}\int \frac{\cos ax}{x^{m-1}}\,dx.$

388. $\int \frac{\cos ax}{x^m}\,dx = \frac{\cos ax}{(1-m)x^{m-1}} + \frac{a}{1-m}\int \frac{\sin ax}{x^{m-1}}\,dx.$

389. $\int \frac{x}{1\pm\sin ax}\,dx = \mp\frac{x\cos ax}{a(1\pm\sin ax)} + \frac{1}{a^2}\log(1\pm\sin ax).$

390. $\int \frac{x}{1+\cos ax}\,dx = \frac{x}{a}\tan\frac{ax}{2} + \frac{2}{a^2}\log\cos\frac{ax}{2}.$

391. $\int \frac{x}{1-\cos ax}\,dx = -\frac{x}{a}\cot\frac{ax}{2} + \frac{2}{a^2}\log\sin\frac{ax}{2}.$

392. $\int \frac{x+\sin x}{1+\cos x}\,dx = x\tan\frac{x}{2}.$

393. $\int \frac{x-\sin x}{1-\cos x}\,dx = -x\cot\frac{x}{2}.$

394. $\int \sqrt{1-\cos ax}\,dx = -\frac{2\sin ax}{a\sqrt{1-\cos ax}} = -\frac{2\sqrt{2}}{a}\cos\frac{ax}{2}.$

395. $\displaystyle\int \sqrt{1+\cos ax}\,dx = \frac{2\sin ax}{a\sqrt{1+\cos ax}} = \frac{2\sqrt{2}}{a}\sin\frac{ax}{2}.$

For the following six integrals, each k represents an integer.

396. $$\int \sqrt{1+\sin x}\,dx = \begin{cases} 2\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right), & (8k-1)\frac{\pi}{2} < x \le (8k+3)\frac{\pi}{2}, \\ \text{or} & \\ -2\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right), & (8k+3)\frac{\pi}{2} < x \le (8k+7)\frac{\pi}{2}. \end{cases}$$

397. $$\int \sqrt{1-\sin x}\,dx = \begin{cases} 2\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right), & (8k-3)\frac{\pi}{2} < x \le (8k+1)\frac{\pi}{2}, \\ \text{or} & \\ -2\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right), & (8k+1)\frac{\pi}{2} < x \le (8k+5)\frac{\pi}{2}. \end{cases}$$

398. $$\int \frac{dx}{\sqrt{1-\cos x}} = \begin{cases} \sqrt{2}\log\tan\frac{x}{4}, & 4k\pi < x \le (4k+2)\pi, \\ \text{or} & \\ -\sqrt{2}\log\tan\frac{x}{4}, & (4k+2)\pi < x \le (4k+4)\pi. \end{cases}$$

399. $$\int \frac{dx}{\sqrt{1+\cos x}} = \begin{cases} \sqrt{2}\log\tan\left(\frac{x+\pi}{4}\right), & (4k-1)\pi < x \le (4k+1)\pi, \\ \text{or} & \\ -\sqrt{2}\log\tan\left(\frac{x+\pi}{4}\right), & (4k+1)\pi < x \le (4k+3)\pi. \end{cases}$$

400. $$\int \frac{dx}{\sqrt{1-\sin x}} = \begin{cases} \sqrt{2}\log\tan\left(\frac{x}{4} - \frac{\pi}{8}\right), & (8k+1)\frac{\pi}{2} < x \le (8k+5)\frac{\pi}{2}, \\ \text{or} & \\ -\sqrt{2}\log\tan\left(\frac{x}{4} - \frac{\pi}{8}\right), & (8k+5)\frac{\pi}{2} < x \le (8k+9)\frac{\pi}{2}. \end{cases}$$

401. $$\int \frac{dx}{\sqrt{1+\sin x}} = \begin{cases} \sqrt{2}\log\tan\left(\frac{x}{4} + \frac{\pi}{8}\right), & (8k-1)\frac{\pi}{2} < x \le (8k+3)\frac{\pi}{2}, \\ \text{or} & \\ -\sqrt{2}\log\tan\left(\frac{x}{4} + \frac{\pi}{8}\right), & (8k+3)\frac{\pi}{2} < x \le (8k+7)\frac{\pi}{2}. \end{cases}$$

402. $\displaystyle\int \tan^2 ax\,dx = \frac{1}{a}\tan ax - x.$

403. $\displaystyle\int \tan^3 ax\,dx = \frac{1}{2a}\tan^2 ax + \frac{1}{a}\log\cos ax.$

404. $\displaystyle\int \tan^4 ax\,dx = \frac{1}{3a}\tan^3 ax - \frac{1}{a}\tan ax + x.$

405. $\displaystyle\int \tan^n ax\,dx = \frac{1}{a(n-1)}\tan^{n-1} ax - \int \tan^{n-2} ax\,dx.$

406. $\displaystyle\int \cot^2 ax\,dx = -\frac{1}{a}\cot ax - x.$

407. $\displaystyle\int \cot^3 ax\,dx = -\frac{1}{2a}\cot^2 ax - \frac{1}{a}\log\sin ax.$

408. $\displaystyle\int \cot^4 ax\,dx = -\frac{1}{3a}\cot^3 ax + \frac{1}{a}\cot ax + x.$

409. $\displaystyle\int \cot^n ax\,dx = -\frac{1}{a(n-1)}\cot^{n-1} ax - \int \cot^{n-2} ax\,dx.$

410. $\displaystyle\int \frac{x}{\sin^2 ax}\,dx = \int x\csc^2 ax\,dx = -\frac{x\cot ax}{a} + \frac{1}{a^2}\log\sin ax.$

411. $$\int \frac{x}{\sin^n ax}\,dx = \int x\csc^n ax\,dx = -\frac{x\cos ax}{a(n-1)\sin^{n-1} ax} - \frac{1}{a^2(n-1)(n-2)\sin^{n-2} ax} + \frac{n-2}{n-1}\int \frac{x}{\sin^{n-2} ax}\,dx.$$

412. $$\int \frac{x}{\cos^2 ax}\,dx = \int x\sec^2 ax\,dx = \frac{x}{a}\tan ax + \frac{1}{a^2}\log\cos ax.$$

413. $$\int \frac{x}{\cos^n ax}\,dx = \int x\sec^n ax\,dx = \frac{x\sin ax}{a(n-1)\cos^{n-1} ax} - \frac{1}{a^2(n-1)(n-2)\cos^{n-2} ax} + \frac{n-2}{n-1}\int \frac{x}{\cos^{n-2} ax}\,dx.$$

414. $$\int \frac{\sin ax}{\sqrt{1+b^2\sin^2 ax}}\,dx = -\frac{1}{ab}\sin^{-1}\frac{b\cos ax}{\sqrt{1+b^2}}.$$

415. $$\int \frac{\sin ax}{\sqrt{1-b^2\sin^2 ax}}\,dx = -\frac{1}{ab}\log\left(b\cos ax + \sqrt{1-b^2\sin^2 ax}\right).$$

416. $$\int (\sin ax)\sqrt{1+b^2\sin^2 ax}\,dx = -\frac{\cos ax}{2a}\sqrt{1+b^2\sin^2 ax} - \frac{1+b^2}{2ab}\sin^{-1}\frac{b\cos ax}{\sqrt{1+b^2}}.$$

417. $$\int (\sin ax)\sqrt{1-b^2\sin^2 ax}\,dx = -\frac{\cos ax}{2a}\sqrt{1-b^2\sin^2 ax} - \frac{1-b^2}{2ab}\log\left(b\cos ax + \sqrt{1-b^2\sin^2 ax}\right).$$

418. $$\int \frac{\cos ax}{\sqrt{1+b^2\sin^2 ax}}\,dx = \frac{1}{ab}\log\left(b\sin ax + \sqrt{1+b^2\sin^2 ax}\right).$$

419. $$\int \frac{\cos ax}{\sqrt{1-b^2\sin^2 ax}}\,dx = \frac{1}{ab}\sin^{-1}(b\sin ax).$$

420. $$\int (\cos ax)\sqrt{1+b^2\sin^2 ax}\,dx = \frac{\sin ax}{2a}\sqrt{1+b^2\sin^2 ax} + \frac{1}{2ab}\log\left(b\sin ax + \sqrt{1+b^2\sin^2 ax}\right).$$

421. $$\int (\cos ax)\sqrt{1-b^2\sin^2 ax}\,dx = \frac{\sin ax}{2a}\sqrt{1-b^2\sin^2 ax} + \frac{1}{2ab}\sin^{-1}(b\sin ax).$$

For the following integral, k represents an integer and $a > |b|$

422. $$\int \frac{dx}{\sqrt{a+b\tan^2 cx}} = \begin{cases} \dfrac{1}{c\sqrt{a-b}}\sin^{-1}\left(\sqrt{\dfrac{a-b}{a}}\sin cx\right), & (4k-1)\frac{\pi}{2} < x \le (4k+1)\frac{\pi}{2}, \\ \text{or} & \\ \dfrac{-1}{c\sqrt{a-b}}\sin^{-1}\left(\sqrt{\dfrac{a-b}{a}}\sin cx\right), & (4k+1)\frac{\pi}{2} < x \le (4k+3)\frac{\pi}{2}. \end{cases}$$

423. $$\int \cos^n x\,dx = \frac{1}{2^{n-1}}\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\frac{\sin[(n-2k)x]}{(n-2k)} + \frac{1}{2^n}\binom{n}{\frac{n}{2}}x, \quad n \text{ is an even integer.}$$

424. $$\int \cos^n x\,dx = \frac{1}{2^{n-1}}\sum_{k=0}^{\frac{n-1}{2}}\binom{n}{k}\frac{\sin[(n-2k)x]}{(n-2k)}, \quad n \text{ is an odd integer.}$$

425. $$\int \sin^n x\,dx = \frac{1}{2^{n-1}}\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\frac{\sin\left(\left[(n-2k)(\frac{\pi}{2}-x)\right]\right)}{(2k-n)} + \frac{1}{2^n}\binom{n}{\frac{n}{2}}x,$$
n is an even integer.

426. $\int \sin^n x\,dx = \frac{1}{2^{n-1}} \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} \frac{\sin\left([(n-2k)(\frac{\pi}{2}-x)]\right)}{(2k-n)}$, n is an odd integer.

5.4.17 FORMS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

427. $\int \sin^{-1} ax\,dx = x\sin^{-1} ax + \frac{\sqrt{1-a^2x^2}}{a}$.

428. $\int \cos^{-1} ax\,dx = x\cos^{-1} ax - \frac{\sqrt{1-a^2x^2}}{a}$.

429. $\int \tan^{-1} ax\,dx = x\tan^{-1} ax - \frac{1}{2a}\log(1+a^2x^2)$.

430. $\int \cot^{-1} ax\,dx = x\cot^{-1} ax + \frac{1}{2a}\log(1+a^2x^2)$.

431. $\int \sec^{-1} ax\,dx = x\sec^{-1} ax - \frac{1}{a}\log\left(ax+\sqrt{a^2x^2-1}\right)$.

432. $\int \csc^{-1} ax\,dx = x\csc^{-1} ax + \frac{1}{a}\log\left(ax+\sqrt{a^2x^2-1}\right)$.

433. $\int \left(\sin^{-1}\frac{x}{a}\right)dx = x\sin^{-1}\frac{x}{a} + \sqrt{a^2-x^2}$, $a>0$.

434. $\int \left(\cos^{-1}\frac{x}{a}\right)dx = x\cos^{-1}\frac{x}{a} - \sqrt{a^2-x^2}$, $a>0$.

435. $\int \left(\tan^{-1}\frac{x}{a}\right)dx = x\tan^{-1}\frac{x}{a} - \frac{a}{2}\log(a^2+x^2)$.

436. $\int \left(\cot^{-1}\frac{x}{a}\right)dx = x\cot^{-1}\frac{x}{a} + \frac{a}{2}\log(a^2+x^2)$.

437. $\int x\sin^{-1}(ax)\,dx = \frac{1}{4a^2}\left((2a^2x^2-1)\sin^{-1}(ax) + ax\sqrt{1-a^2x^2}\right)$.

438. $\int x\cos^{-1}(ax)\,dx = \frac{1}{4a^2}\left((2a^2x^2-1)\cos^{-1}(ax) - ax\sqrt{1-a^2x^2}\right)$.

439. $\int x^n\sin^{-1}(ax)\,dx = \frac{x^{n+1}}{n+1}\sin^{-1}(ax) - \frac{a}{n+1}\int\frac{x^{n+1}}{\sqrt{1-a^2x^2}}\,dx$, $n \neq -1$.

440. $\int x^n\cos^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\cos^{-1}(ax) + \frac{a}{n+1}\int\frac{x^{n+1}}{\sqrt{1-a^2x^2}}\,dx$, $n \neq -1$.

441. $\int x\tan^{-1}(ax)\,dx = \frac{1+a^2x^2}{2a^2}\tan^{-1}(ax) - \frac{x}{2a}$.

442. $\int x^n\tan^{-1}(ax)\,dx = \frac{x^{n+1}}{n+1}\tan^{-1}(ax) - \frac{a}{n+1}\int\frac{x^{n+1}}{1+a^2x^2}\,dx$.

443. $\int x\cot^{-1}(ax)\,dx = \frac{1+a^2x^2}{2a^2}\cot^{-1}(ax) + \frac{x}{2a}$.

444. $\int x^n\cot^{-1}(ax)\,dx = \frac{x^{n+1}}{n+1}\cot^{-1}(ax) + \frac{a}{n+1}\int\frac{x^{n+1}}{1+a^2x^2}\,dx$.

445. $\int \frac{\sin^{-1}(ax)}{x^2}\,dx = a\log\left(\frac{1-\sqrt{1-a^2x^2}}{x}\right) - \frac{\sin^{-1}(ax)}{x}$.

446. $\int \frac{\cos^{-1}(ax)}{x^2}\,dx = -\frac{1}{x}\cos^{-1}(ax) + a\log\left(\frac{1+\sqrt{1-a^2x^2}}{x}\right)$.

447. $$\int \frac{\tan^{-1}(ax)}{x^2}\,dx = -\frac{1}{x}\tan^{-1}(ax) - \frac{a}{2}\log\left(\frac{1+a^2x^2}{x^2}\right).$$

448. $$\int \frac{\cot^{-1}(ax)}{x^2}\,dx = -\frac{1}{x}\cot^{-1}(ax) - \frac{a}{2}\log\left(\frac{x^2}{1+a^2x^2}\right).$$

449. $$\int (\sin^{-1}(ax))^2\,dx = x(\sin^{-1}(ax))^2 - 2x + \frac{2\sqrt{1-a^2x^2}}{a}\sin^{-1}(ax).$$

450. $$\int (\cos^{-1}(ax))^2\,dx = x(\cos^{-1}(ax))^2 - 2x - \frac{2\sqrt{1-a^2x^2}}{a}\cos^{-1}(ax).$$

451. $$\int (\sin^{-1}(ax))^n\,dx = \begin{cases} x(\sin^{-1}(ax))^n + \dfrac{n\sqrt{1-a^2x^2}}{a}(\sin^{-1}(ax))^{n-1} - n(n-1)\displaystyle\int (\sin^{-1}(ax))^{n-2}\,dx, \\ \quad \text{or} \\ \displaystyle\sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^r n!}{(n-2r)!} x(\sin^{-1} ax)^{n-2r} + \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^r \frac{n!\sqrt{1-a^2x^2}}{(n-2r-1)!a}(\sin^{-1} ax)^{n-2r-1}. \end{cases}$$

452. $$\int (\cos^{-1}(ax))^n\,dx = \begin{cases} x(\cos^{-1}(ax))^n - \dfrac{n\sqrt{1-a^2x^2}}{a}(\cos^{-1}(ax))^{n-1} - n(n-1)\displaystyle\int (\cos^{-1}(ax))^{n-2}\,dx, \\ \quad \text{or} \\ \displaystyle\sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^r n!}{(n-2r)!} x(\cos^{-1} ax)^{n-2r} - \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^r \frac{n!\sqrt{1-a^2x^2}}{(n-2r-1)!a}(\cos^{-1} ax)^{n-2r-1}. \end{cases}$$

453. $$\int \frac{\sin^{-1} ax}{\sqrt{1-a^2x^2}}\,dx = \frac{1}{2a}\left(\sin^{-1} ax\right)^2.$$

454. $$\int \frac{x^n \sin^{-1} ax}{\sqrt{1-a^2x^2}}\,dx = -\frac{x^{n-1}}{na^2}\sqrt{1-a^2x^2}\sin^{-1} ax + \frac{x^n}{n^2a} + \frac{n-1}{na^2}\int \frac{x^{n-2}\sin^{-1} ax}{\sqrt{1-a^2x^2}}\,dx.$$

455. $$\int \frac{\cos^{-1} ax}{\sqrt{1-a^2x^2}}\,dx = -\frac{1}{2a}\left(\cos^{-1} ax\right)^2.$$

456. $$\int \frac{x^n \cos^{-1} ax}{\sqrt{1-a^2x^2}}\,dx = -\frac{x^{n-1}}{na^2}\sqrt{1-a^2x^2}\cos^{-1} ax - \frac{x^n}{n^2a} + \frac{n-1}{na^2}\int \frac{x^{n-2}\cos^{-1} ax}{\sqrt{1-a^2x^2}}\,dx.$$

457. $$\int \frac{\tan^{-1} ax}{1+a^2x^2}\,dx = \frac{1}{2a}\left(\tan^{-1} ax\right)^2.$$

458. $$\int \frac{\cot^{-1} ax}{1+a^2x^2}\,dx = -\frac{1}{2a}\left(\cot^{-1} ax\right)^2.$$

459. $$\int x\sec^{-1} ax\,dx = \frac{x^2}{2}\sec^{-1} ax - \frac{1}{2a^2}\sqrt{a^2x^2-1}.$$

460. $$\int x^n\sec^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\sec^{-1} ax - \frac{1}{n+1}\int \frac{x^n}{\sqrt{a^2x^2-1}}\,dx.$$

461. $$\int \frac{\sec^{-1} ax}{x^2}\,dx = -\frac{\sec^{-1} ax}{x} + \frac{\sqrt{a^2x^2-1}}{x}.$$

462. $\displaystyle\int x\csc^{-1} ax\,dx = \frac{x^2}{2}\csc^{-1} ax + \frac{1}{2a^2}\sqrt{a^2x^2-1}.$

463. $\displaystyle\int x^n\csc^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\csc^{-1} ax + \frac{1}{n+1}\int \frac{x^n}{\sqrt{a^2x^2-1}}\,dx.$

464. $\displaystyle\int \frac{\csc^{-1} ax}{x^2}\,dx = -\frac{\csc^{-1} ax}{x} - \frac{\sqrt{a^2x^2-1}}{x}.$

5.4.18 LOGARITHMIC FORMS

465. $\displaystyle\int \log x\,dx = x\log x - x.$

466. $\displaystyle\int x\log x\,dx = \frac{x^2}{2}\log x - \frac{x^2}{4}.$

467. $\displaystyle\int x^2\log x\,dx = \frac{x^3}{3}\log x - \frac{x^3}{9}.$

468. $\displaystyle\int x^n\log x\,dx = \frac{x^{n+1}}{n+1}\log x - \frac{x^{n+1}}{(n+1)^2}.$

469. $\displaystyle\int (\log x)^2\,dx = x(\log x)^2 - 2x\log x + 2x.$

470. $$\int (\log x)^n\,dx = \begin{cases} x(\log x)^n - n\displaystyle\int (\log x)^{n-1}\,dx, & n \neq -1, \\ \text{or} & \\ (-1)^n n!x\displaystyle\sum_{r=0}^{n}\frac{(-\log x)^r}{r!}, & n \neq -1. \end{cases}$$

471. $\displaystyle\int \frac{(\log x)^n}{x}\,dx = \frac{1}{n+1}(\log x)^{n+1}, \quad n \neq -1.$

472. $\displaystyle\int \frac{dx}{\log x} = \log(\log x) + \log x + \frac{(\log x)^2}{2\cdot 2!} + \frac{(\log x)^3}{3\cdot 3!} + \dots.$

473. $\displaystyle\int \frac{dx}{x\log x} = \log(\log x).$

474. $\displaystyle\int \frac{dx}{x(\log x)^n} = \frac{1}{(1-n)(\log x)^{n-1}}, \quad n \neq 1.$

475. $\displaystyle\int \frac{x^m\,dx}{(\log x)^n} = \frac{x^{m+1}}{(1-n)(\log x)^{n-1}} + \frac{m+1}{n-1}\int \frac{x^m\,dx}{(\log x)^{n-1}}, \quad n \neq 1.$

476. $$\int x^m(\log x)^n\,dx = \begin{cases} \dfrac{x^{m+1}(\log x)^n}{m+1} - \dfrac{n}{m+1}\displaystyle\int x^m(\log x)^{n-1}\,dx, \\ \text{or} \\ (-1)^n\dfrac{n!}{m+1}x^{m+1}\displaystyle\sum_{r=0}^{n}\frac{(-\log x)^r}{r!(m+1)^{n-r}}. \end{cases}$$

477. $\displaystyle\int x^p\cos(b\log x)\,dx = \frac{x^{p+1}}{(p+1)^2+b^2}\left[b\sin(b\log x) + (p+1)\cos(b\log x)\right].$

478. $\displaystyle\int x^p\sin(b\log x)\,dx = \frac{x^{p+1}}{(p+1)^2+b^2}\left[(p+1)\sin(b\log x) - b\cos(b\log x)\right].$

479. $\displaystyle\int \log(ax+b)\,dx = \frac{ax+b}{a}\log(ax+b) - x.$

480. $\displaystyle\int \frac{\log(ax+b)}{x^2}\,dx = \frac{a}{b}\log x - \frac{ax+b}{bx}\log(ax+b).$

481. $\displaystyle\int x^m \log(ax+b)\,dx = \frac{1}{m+1}\left[x^{m+1} - \left(-\frac{b}{a}\right)^{m+1}\right]\log(ax+b)$
$$-\frac{1}{m+1}\left(-\frac{b}{a}\right)^{m+1}\sum_{r=1}^{m+1}\frac{1}{r}\left(-\frac{ax}{b}\right)^r.$$

482. $\displaystyle\int \frac{\log(ax+b)}{x^m}\,dx = -\frac{1}{m-1}\frac{\log(ax+b)}{x^{m-1}} + \frac{1}{m-1}\left(-\frac{a}{b}\right)^{m-1}\log\frac{ax+b}{x}$
$$+\frac{1}{m-1}\left(-\frac{a}{b}\right)^{m-1}\sum_{r=1}^{m-2}\frac{1}{r}\left(-\frac{b}{ax}\right)^r, \quad m>2.$$

483. $\displaystyle\int \log\frac{x+a}{x-a}\,dx = (x+a)\log(x+a) - (x-a)\log(x-a).$

484. $\displaystyle\int x^m \log\frac{x+a}{x-a}\,dx = \frac{x^{m+1}-(-a)^{m+1}}{m+1}\log(x+a) - \frac{x^{m+1}-a^{m+1}}{m+1}\log(x-a)$
$$+\frac{2a^{m+1}}{m+1}\sum_{r=1}^{\lfloor\frac{m+1}{2}\rfloor}\frac{1}{m-2r+2}\left(\frac{x}{a}\right)^{m-2r+2}.$$

485. $\displaystyle\int \frac{1}{x^2}\log\frac{x+a}{x-a}\,dx = \frac{1}{x}\log\frac{x-a}{x+a} - \frac{1}{a}\log\frac{x^2-a^2}{x^2}.$

For the following two integrals, $X = a + bx + cx^2$.

486. $\displaystyle\int \log X\,dx =$
$$\begin{cases} \left(x+\dfrac{b}{2c}\right)\log X - 2x + \dfrac{\sqrt{4ac-b^2}}{c}\tan^{-1}\dfrac{2cx+b}{\sqrt{4ac-b^2}}, & b^2-4ac<0, \\ \text{or} & \\ \left(x+\dfrac{b}{2c}\right)\log X - 2x + \dfrac{\sqrt{b^2-4ac}}{c}\tanh^{-1}\dfrac{2cx+b}{\sqrt{b^2-4ac}}, & b^2-4ac>0. \end{cases}$$

487. $\displaystyle\int x^n \log X\,dx = \frac{x^{n+1}}{n+1}\log X - \frac{2c}{n+1}\int\frac{x^{n+2}}{X}\,dx - \frac{b}{n+1}\int\frac{x^{n+1}}{X}\,dx, \quad n\neq -1.$

488. $\displaystyle\int \log(x^2+a^2)\,dx = x\log(x^2+a^2) - 2x + 2a\tan^{-1}\frac{x}{a}.$

489. $\displaystyle\int \log(x^2-a^2)\,dx = x\log(x^2-a^2) - 2x + a\log\frac{x+a}{x-a}.$

490. $\displaystyle\int x\log(x^2+a^2)\,dx = \frac{1}{2}(x^2+a^2)\log(x^2+a^2) - \frac{1}{2}x^2.$

491. $\displaystyle\int \log\left(x+\sqrt{x^2\pm a^2}\right)dx = x\log\left(x+\sqrt{x^2\pm a^2}\right) - \sqrt{x^2\pm a^2}.$

492. $\displaystyle\int x\log\left(x+\sqrt{x^2\pm a^2}\right)dx = \left(\frac{x^2}{2}\pm\frac{a^2}{4}\right)\log\left(x+\sqrt{x^2\pm a^2}\right) - \frac{x\sqrt{x^2\pm a^2}}{4}.$

493. $\displaystyle\int x^m\log\left(x+\sqrt{x^2\pm a^2}\right)dx = \frac{x^{m+1}}{m+1}\log\left(x+\sqrt{x^2\pm a^2}\right)$
$$-\frac{1}{m+1}\int\frac{x^{m+1}}{\sqrt{x^2\pm a^2}}\,dx.$$

494. $\displaystyle\int \frac{\log\left(x+\sqrt{x^2+a^2}\right)}{x^2}\,dx = -\frac{\log\left(x+\sqrt{x^2+a^2}\right)}{x} - \frac{1}{a}\log\frac{a+\sqrt{x^2+a^2}}{x}.$

495. $\displaystyle\int \frac{\log\left(x+\sqrt{x^2-a^2}\right)}{x^2}\,dx = -\frac{\log\left(x+\sqrt{x^2-a^2}\right)}{x} + \frac{1}{|a|}\sec^{-1}\frac{x}{a}.$

496. $$\int x^n \log\left(x^2-a^2\right)dx = \frac{1}{n+1}\left[x^{n+1}\log\left(x^2-a^2\right) - a^{n+1}\log(x-a) - (-a)^{n+1}\log(x+a) - 2\sum_{r=0}^{\lfloor \frac{n}{2}\rfloor}\frac{a^{2r}x^{n-2r+1}}{n-2r+1}\right].$$

5.4.19 EXPONENTIAL FORMS

497. $\displaystyle\int e^x\,dx = e^x.$

498. $\displaystyle\int e^{-x}\,dx = -e^{-x}.$

499. $\displaystyle\int e^{ax}\,dx = \frac{e^{ax}}{a}.$

500. $\displaystyle\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1).$

501. $$\int x^m e^{ax}\,dx = \begin{cases} \dfrac{x^m e^{ax}}{a} - \dfrac{m}{a}\displaystyle\int x^{m-1}e^{ax}\,dx, \\ \text{or} \\ e^{ax}\displaystyle\sum_{r=0}^{m}(-1)^r\frac{m!x^{m-r}}{(m-r)!a^{r+1}}. \end{cases}$$

502. $\displaystyle\int \frac{e^{ax}}{x}\,dx = \log x + \frac{ax}{1!} + \frac{a^2x^2}{2\cdot 2!} + \frac{a^3x^3}{3\cdot 3!} + \dots.$

503. $\displaystyle\int \frac{e^{ax}}{x^m}\,dx = \frac{1}{1-m}\frac{e^{ax}}{x^{m-1}} + \frac{a}{m-1}\int\frac{e^{ax}}{x^{m-1}}\,dx, \quad m\neq 1.$

504. $\displaystyle\int e^{ax}\log x\,dx = \frac{e^{ax}\log x}{a} - \frac{1}{a}\int\frac{e^{ax}}{x}\,dx.$

505. $\displaystyle\int \frac{dx}{1+e^x} = x - \log\left(1+e^x\right) = \log\frac{e^x}{1+e^x}.$

506. $\displaystyle\int \frac{dx}{a+be^{px}} = \frac{x}{a} - \frac{1}{ap}\log\left(a+be^{px}\right).$

507. $\displaystyle\int \frac{dx}{ae^{mx}+be^{-mx}} = \frac{1}{m\sqrt{ab}}\tan^{-1}\left(e^{mx}\sqrt{\frac{a}{b}}\right), \quad a>0,\ b>0.$

508. $$\int \frac{dx}{ae^{mx}-be^{-mx}} = \begin{cases} \dfrac{1}{2m\sqrt{ab}}\log\left(\dfrac{\sqrt{a}e^{mx}-\sqrt{b}}{\sqrt{a}e^{mx}+\sqrt{b}}\right), & a>0,\ b>0, \\ \text{or} \\ \dfrac{-1}{m\sqrt{ab}}\tanh^{-1}\left(\sqrt{\dfrac{a}{b}}e^{mx}\right), & a>0,\ b>0. \end{cases}$$

509. $\displaystyle\int \left(a^x - a^{-x}\right)dx = \frac{a^x+a^{-x}}{\log a}.$

510. $\displaystyle\int \frac{e^{ax}}{b+ce^{ax}}\,dx = \frac{1}{ac}\log\left(b+ce^{ax}\right).$

511. $\displaystyle\int \frac{xe^{ax}}{(1+ax)^2}\,dx = \frac{e^{ax}}{a^2(1+ax)}.$

512. $$\int xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2}.$$

513. $$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}\left[a\sin(bx) - b\cos(bx)\right]}{a^2+b^2}.$$

514. $$\int e^{ax}\sin(bx)\sin(cx)\,dx = \frac{e^{ax}\left[(b-c)\sin(b-c)x + a\cos(b-c)x\right]}{2\left[a^2+(b-c)^2\right]} - \frac{e^{ax}\left[(b+c)\sin(b+c)x + a\cos(b+c)x\right]}{2\left[a^2+(b+c)^2\right]}.$$

515. $$\int e^{ax}\sin(bx)\cos(cx)\,dx = \frac{e^{ax}\left[a\sin(b-c)x - (b-c)\cos(b-c)x\right]}{2\left[a^2+(b-c)^2\right]} + \frac{e^{ax}\left[a\sin(b+c)x - (b+c)\cos(b+c)x\right]}{2\left[a^2+(b+c)^2\right]}.$$

516. $$\int e^{ax}\sin(bx)\sin(bx+c)\,dx = \frac{e^{ax}\cos c}{2a} - \frac{e^{ax}\left[a\cos 2bx + c + 2b\sin 2bx + c\right]}{2\left[a^2+4b^2\right]}.$$

517. $$\int e^{ax}\sin(bx)\cos(bx+c)\,dx = -\frac{e^{ax}\sin c}{2a} + \frac{e^{ax}\left[a\sin 2bx + c - 2b\cos 2bx + c\right]}{2\left[a^2+4b^2\right]}.$$

518. $$\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx) + b\sin(bx)\right].$$

519. $$\int e^{ax}\cos(bx)\cos(cx)\,dx = \frac{e^{ax}\left[(b-c)\sin(b-c)x + a\cos(b-c)x\right]}{2\left[a^2+(b-c)^2\right]} + \frac{e^{ax}\left[(b+c)\sin(b+c)x + a\cos(b+c)x\right]}{2\left[a^2+(b+c)^2\right]}.$$

520. $$\int e^{ax}\cos(bx)\cos(bx+c)\,dx = \frac{e^{ax}\cos c}{2a} + \frac{e^{ax}\left[a\cos 2bx + c + 2b\sin 2bx + c\right]}{2\left[a^2+4b^2\right]}.$$

521. $$\int e^{ax}\cos(bx)\sin(bx+c)\,dx = \frac{e^{ax}\sin c}{2a} + \frac{e^{ax}\left[a\sin 2bx + c - 2b\cos 2bx + c\right]}{2\left[a^2+4b^2\right]}.$$

522. $$\int e^{ax}\sin^n(bx)\,dx = \frac{1}{a^2+n^2b^2}\left[(a\sin(bx) - nb\cos(bx))e^{ax}\sin^{n-1}(bx) + n(n-1)b^2\int e^{ax}\sin^{n-2}(bx)\,dx\right].$$

523. $$\int e^{ax}\cos^n(bx)\,dx = \frac{1}{a^2+n^2b^2}\left[(a\cos(bx) + nb\sin(bx))e^{ax}\cos^{n-1}(bx) + n(n-1)b^2\int e^{ax}\cos^{n-2}(bx)\,dx\right].$$

524. $$\int x^m e^x\sin x\,dx = \frac{1}{2}x^m e^x(\sin x - \cos x) - \frac{m}{2}\int x^{m-1}e^x\sin x\,dx + \frac{m}{2}\int x^{m-1}e^x\cos x\,dx.$$

525. $$\int x^m e^{ax}\sin bx\,dx = x^m e^{ax}\frac{a\sin(bx) - b\cos(bx)}{a^2+b^2} - \frac{m}{a^2+b^2}\int x^{m-1}e^{ax}(a\sin(bx) - b\cos(bx))\,dx.$$

526. $$\int x^m e^x\cos x\,dx = \frac{1}{2}x^m e^x(\sin x + \cos x) - \frac{m}{2}\int x^{m-1}e^x\sin x\,dx - \frac{m}{2}\int x^{m-1}e^x\cos x\,dx.$$

527. $$\int x^m e^{ax}\cos bx\,dx = x^m e^{ax}\frac{a\cos(bx) + b\sin(bx)}{a^2+b^2} - \frac{m}{a^2+b^2}\int x^{m-1}e^{ax}(a\cos(bx) + b\sin(bx))\,dx.$$

528. $\int e^{ax}\cos^m x \sin^n x\,dx =$

$$\begin{cases} \dfrac{e^{ax}(\cos^{m-1} x)(\sin^n x)\left[a\cos x + (m+n)\sin x\right]}{(m+n)^2+a^2} \\ \qquad -\dfrac{na}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^{m-1} x)(\sin^{n-1} x)\,dx \\ \qquad +\dfrac{(m-1)(m+n)}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^{m-2} x)(\sin^n x)\,dx, \\ \text{or} \\ \dfrac{e^{ax}(\cos^m x)(\sin^{n-1} x)\left[a\sin x - (m+n)\cos x\right]}{(m+n)^2+a^2} \\ \qquad +\dfrac{ma}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^{m-1} x)(\sin^{n-1} x)\,dx \\ \qquad +\dfrac{(n-1)(m+n)}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^m x)(\sin^{n-2} x)\,dx, \\ \text{or} \\ \dfrac{e^{ax}(\cos^{m-1} x)(\sin^{n-1} x)\left[a\sin x\cos x + m\sin^2 x - n\cos^2 x\right]}{(m+n)^2+a^2} \\ \qquad +\dfrac{m(m-1)}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^{m-2} x)(\sin^n x)\,dx \\ \qquad +\dfrac{n(n-1)}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^m x)(\sin^{n-2} x)\,dx, \\ \text{or} \\ \dfrac{e^{ax}(\cos^{m-1} x)(\sin^{n-1} x)\left[a\sin x\cos x + m\sin^2 x - n\cos^2 x\right]}{(m+n)^2+a^2} \\ \qquad +\dfrac{m(m-1)}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^{m-2} x)(\sin^{n-2} x)\,dx \\ \qquad +\dfrac{(n-m)(n+m-1)}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^m x)(\sin^{n-2} x)\,dx. \end{cases}$$

529. $$\int xe^{ax}\sin(bx)\,dx = \frac{xe^{ax}}{a^2+b^2}\left[a\sin(bx) - b\cos(bx)\right] - \frac{e^{ax}}{(a^2+b^2)^2}\left[\left(a^2-b^2\right)\sin bx - 2ab\cos(bx)\right].$$

530. $$\int xe^{ax}\cos(bx)\,dx = \frac{xe^{ax}}{a^2+b^2}\left[a\cos(bx) + b\sin(bx)\right] - \frac{e^{ax}}{(a^2+b^2)^2}\left[\left(a^2-b^2\right)\cos bx + 2ab\sin(bx)\right].$$

531. $$\int \frac{e^{ax}}{\sin^n x}\,dx = -\frac{e^{ax}\left[a\sin x + (n-2)\cos x\right]}{(n-1)(n-2)\sin^{n-1} x} + \frac{a^2+(n-2)^2}{(n-1)(n-2)}\int \frac{e^{ax}}{\sin^{n-2} x}\,dx.$$

532. $$\int \frac{e^{ax}}{\cos^n x}\,dx = -\frac{e^{ax}\left[a\cos x - (n-2)\sin x\right]}{(n-1)(n-2)\cos^{n-1} x} + \frac{a^2+(n-2)^2}{(n-1)(n-2)}\int \frac{e^{ax}}{\cos^{n-2} x}\,dx.$$

533. $$\int e^{ax}\tan^n x\,dx = e^{ax}\frac{\tan^{n-1} x}{n-1} - \frac{a}{n-1}\int e^{ax}\tan^{n-1} x\,dx - \int e^{ax}\tan^{n-2} x\,dx.$$

5.4.20 HYPERBOLIC FORMS

534. $\int \sinh x\,dx = \cosh x.$

535. $\int \cosh x\,dx = \sinh x.$

536. $\int \tanh x\,dx = \log\cosh x.$

537. $\int \coth x\,dx = \log\sinh x.$

538. $\int \operatorname{sech} x\,dx = \tan^{-1}(\sinh x).$

539. $\int \operatorname{csch} x\,dx = \log\tanh\left(\frac{x}{2}\right).$

540. $\int x\sinh x\,dx = x\cosh x - \sinh x.$

541. $\int x^n\sinh x\,dx = x^n\cosh x - n\int x^{n-1}(\cosh x)\,dx.$

542. $\int x\cosh x\,dx = x\sinh x - \cosh x.$

543. $\int x^n\cosh x\,dx = x^n\sinh x - n\int x^{n-1}(\sinh x)\,dx.$

544. $\int \operatorname{sech} x\tanh x\,dx = -\operatorname{sech} x.$

545. $\int \operatorname{csch} x\coth x\,dx = -\operatorname{csch} x.$

546. $\int \sinh^2 x\,dx = \frac{\sinh 2x}{4} - \frac{x}{2}.$

547. $\int \sinh^m x\cosh^n x\,dx =$

$$\begin{cases} \frac{1}{m+n}\sinh^{m+1}x\cosh^{n-1}x + \frac{n-1}{m+n}\int \sinh^m x\cosh^{n-2}x\,dx, & m+n\neq 0, \\ \text{or} & \\ \frac{1}{m+n}\sinh^{m-1}x\cosh^{n+1}x - \frac{m-1}{m+n}\int \sinh^{m-2}x\cosh^n x\,dx, & m+n\neq 0. \end{cases}$$

548. $\int \frac{dx}{(\sinh^m x)(\cosh^n x)} =$

$$\begin{cases} -\frac{1}{(m-1)(\sinh^{m-1}x)(\cosh^{n-1}x)} - \frac{m+n-2}{m-1}\int \frac{dx}{(\sinh^{m-2}x)(\cosh^n x)}\,dx, & m\neq 1, \\ \text{or} & \\ \frac{1}{(n-1)(\sinh^{m-1}x)(\cosh^{n-1}x)} + \frac{m+n-2}{n-1}\int \frac{dx}{(\sinh^m x)(\cosh^{n-2}x)}\,dx, & n\neq 1. \end{cases}$$

549. $\int \tanh^2 x\,dx = x - \tanh x.$

550. $\int \tanh^n x\,dx = -\frac{\tanh^{n-1}x}{n-1} + \int (\tanh^{n-2}x)\,dx, \quad n\neq 1.$

551. $\int \operatorname{sech}^2 x\,dx = \tanh x.$

552. $\int \cosh^2 x\,dx = \frac{\sinh 2x}{4} + \frac{x}{2}.$

553. $\int \coth^2 x\,dx = x - \coth x.$

554. $\int \coth^n x\,dx = -\frac{\coth^{n-1}x}{n-1} + \int \coth^{n-2}x\,dx, \quad n\neq 1.$

555. $\int \operatorname{csch}^2 x\, dx = -\coth x.$

556. $\int (\sinh mx)(\sinh nx)\, dx = \frac{\sinh(m+n)x}{2(m+n)} - \frac{\sinh(m-n)x}{2(m-n)}, \quad m^2 \neq n^2.$

557. $\int (\cosh mx)(\cosh nx)\, dx = \frac{\sinh(m+n)x}{2(m+n)} + \frac{\sinh(m-n)x}{2(m-n)}, \quad m^2 \neq n^2.$

558. $\int (\sinh mx)(\cosh nx)\, dx = \frac{\cosh(m+n)x}{2(m+n)} + \frac{\cosh(m-n)x}{2(m-n)}, \quad m^2 \neq n^2.$

559. $\int \left(\sinh^{-1}\frac{x}{a}\right) dx = x\sinh^{-1}\frac{x}{a} - \sqrt{x^2+a^2}, \quad a > 0.$

560. $\int x\left(\sinh^{-1}\frac{x}{a}\right) dx = \left(\frac{x^2}{2}+\frac{a^2}{4}\right)\sinh^{-1}\frac{x}{a} - \frac{x}{4}\sqrt{x^2+a^2}, \quad a > 0.$

561. $\int x^n \sinh^{-1} x\, dx = \frac{x^{n+1}}{n+1}\sinh^{-1} x - \frac{1}{n+1}\int \frac{x^{n+1}}{\sqrt{1+x^2}}\, dx, \quad n \neq -1.$

562. $\int^z \cosh^{-1}\frac{x}{a}\, dx = \begin{cases} z\cosh^{-1}\frac{z}{a} - \sqrt{z^2-a^2}, & \cosh^{-1}\frac{z}{a} > 0, \\ \text{or} & \\ z\cosh^{-1}\frac{z}{a} + \sqrt{z^2-a^2}, & \cosh^{-1}\frac{z}{a} < 0,\ a > 0. \end{cases}$

563. $\int x\left(\cosh^{-1}\frac{x}{a}\right) dx = \left(\frac{x^2}{2}-\frac{a^2}{4}\right)\cosh^{-1}\frac{x}{a} - \frac{x}{4}\sqrt{x^2-a^2}.$

564. $\int x^n \cosh^{-1} x\, dx = \frac{x^{n+1}}{n+1}\cosh^{-1} x - \frac{1}{n+1}\int \frac{x^{n+1}}{\sqrt{x^2-1}}\, dx, \quad n \neq -1.$

565. $\int \left(\tanh^{-1}\frac{x}{a}\right) dx = x\tanh^{-1}\frac{x}{a} + \frac{a}{2}\log(a^2-x^2), \quad \left|\frac{x}{a}\right| < 1.$

566. $\int \left(\coth^{-1}\frac{x}{a}\right) dx = x\coth^{-1}\frac{x}{a} + \frac{a}{2}\log(x^2-a^2), \quad \left|\frac{x}{a}\right| > 1.$

567. $\int x\left(\tanh^{-1}\frac{x}{a}\right) dx = \frac{x^2-a^2}{2}\tanh^{-1}\frac{x}{a} + \frac{ax}{2}, \quad \left|\frac{x}{a}\right| < 1.$

568. $\int x^n \tanh^{-1} x\, dx = \frac{x^{n+1}}{n+1}\tanh^{-1} x - \frac{1}{n+1}\int \frac{x^{n+1}}{1-x^2}\, dx, \quad n \neq -1.$

569. $\int x\left(\coth^{-1}\frac{x}{a}\right) dx = \frac{x^2-a^2}{2}\coth^{-1}\frac{x}{a} + \frac{ax}{2}, \quad \left|\frac{x}{a}\right| > 1.$

570. $\int x^n \coth^{-1} x\, dx = \frac{x^{n+1}}{n+1}\coth^{-1} x + \frac{1}{n+1}\int \frac{x^{n+1}}{x^2-1}\, dx, \quad n \neq -1.$

571. $\int \operatorname{sech}^{-1} x\, dx = x\operatorname{sech}^{-1} x + \sin^{-1} x.$

572. $\int x\operatorname{sech}^{-1} x\, dx = \frac{x^2}{2}\operatorname{sech}^{-1} x - \frac{1}{2}\sqrt{1-x^2}.$

573. $\int x^n \operatorname{sech}^{-1} x\, dx = \frac{x^{n+1}}{n+1}\operatorname{sech}^{-1} x + \frac{1}{n+1}\int \frac{x^n}{\sqrt{1-x^2}}\, dx, \quad n \neq -1.$

574. $\int \operatorname{csch}^{-1} x\, dx = x\operatorname{csch}^{-1} x + \frac{x}{|x|}\sinh^{-1} x.$

575. $\int x\operatorname{csch}^{-1} x\, dx = \frac{x^2}{2}\operatorname{csch}^{-1} x + \frac{1}{2}\frac{x}{|x|}\sqrt{1+x^2}.$

576. $\displaystyle\int x^n \operatorname{csch}^{-1} x\, dx = \frac{x^{n+1}}{n+1} \operatorname{csch}^{-1} x + \frac{1}{n+1}\frac{x}{|x|}\int \frac{x^n}{\sqrt{1+x^2}}\, dx, \quad n \neq -1.$

5.4.21 BESSEL FUNCTIONS

$Z_p(x)$ represents any of the Bessel functions $\{J_p(x), Y_p(x), K_p(x), I_p(x)\}$.

577. $\displaystyle\int x^{p+1} Z_p(x)\, dx = x^{p+1} Z_{p+1}(x).$

578. $\displaystyle\int x^{-p+1} Z_p(x)\, dx = -x^{-p+1} Z_{p-1}(x).$

579. $\displaystyle\int x\,[Z_p(ax)]^2\, dx = \frac{x^2}{2}\left[[Z_p(ax)]^2 - Z_{p-1}(ax)Z_{p+1}(ax)\right].$

580. $\displaystyle\int Z_1(x)\, dx = -Z_0(x).$

581. $\displaystyle\int xZ_0(x)\, dx = xZ_1(x).$

5.5 TABLE OF DEFINITE INTEGRALS

582. $\displaystyle\int_0^\infty x^{n-1}e^{-x}\, dx = \Gamma(n), \qquad \operatorname{Re} n > 0.$

583. $\displaystyle\int_0^\infty x^n p^{-x}\, dx = \frac{n!}{(\log p)^{n+1}}, \quad p > 0,$ n is a non-negative integer.

584. $\displaystyle\int_0^\infty x^{n-1}e^{-(a+1)x}\, dx = \frac{\Gamma(n)}{(a+1)^n}, \quad n > 0,\ a > -1.$

585. $\displaystyle\int_0^1 x^m \left(\log\frac{1}{x}\right)^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}, \quad m > -1,\ n > -1.$

586. $\displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}\, dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}, \quad n > 0,\ m > 0.$

587. $\displaystyle\int_a^b (x-a)^m(b-x)^n\, dx = (b-a)^{m+n+1}\frac{\Gamma(m+1)\,\Gamma(n+1)}{\Gamma(m+n+2)},$

$m > -1,\ n > -1,\ b > a.$

588. $\displaystyle\int_1^\infty \frac{dx}{x^m} = \frac{1}{m-1}, \quad m > 1.$

589. $\displaystyle\int_0^\infty \frac{dx}{(1+x)x^p} = \pi \csc p\pi, \quad 0 < p < 1.$

590. $\displaystyle\int_0^\infty \frac{dx}{(1-x)x^p} = -\pi \cot p\pi, \quad 0 < p < 1.$

591. $\displaystyle\int_0^1 \frac{x^p}{(1-x)^p}\, dx = p\pi \csc p\pi, \quad |p| < 1.$

592. $\displaystyle\int_0^1 \frac{x^p}{(1-x)^{p+1}}\, dx = \int_0^1 \frac{(1-x)^p}{x^{p+1}}\, dx = -\pi \operatorname{cosec} p\pi, \quad -1 < p < 0.$

593. $\displaystyle\int_0^\infty \frac{x^{p-1}}{1+x}\, dx = \frac{\pi}{\sin p\pi}, \quad 0 < p < 1.$

594. $\int_0^\infty \frac{x^{m-1}}{1+x^n}\,dx = \frac{\pi}{n\sin\frac{m\pi}{n}}, \quad 0 < m < n.$

595. $\int_0^\infty \frac{x^a}{(m+x^b)^c}\,dx = \frac{m^{(a+1-bc)/b}}{b}\frac{\Gamma\left(\frac{a+1}{b}\right)\Gamma\left(c-\frac{a+1}{b}\right)}{\Gamma(c)},$

$a > -1,\ b > 0,\ m > 0,\ c > \frac{a+1}{b}.$

596. $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}} = \pi.$

597. $\int_0^\infty \frac{a}{a^2+x^2}\,dx = \begin{cases} \frac{\pi}{2}, & a > 0, \\ \text{or} & \\ 0, & a = 0, \\ \text{or} & \\ -\frac{\pi}{2}, & a < 0. \end{cases}$

598. $\int_0^a \left(a^2-x^2\right)^{n/2}\,dx = \int_{-a}^a \frac{1}{2}\left(a^2-x^2\right)^{n/2}\,dx = \frac{n!!}{(n+1)!!}\frac{\pi}{2}a^{n+1},$

$a > 0,\ n$ is an odd integer.

599. $\int_0^a x^m\left(a^2-x^2\right)^{n/2}\,dx = \frac{1}{2}a^{m+n+1}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{m+n+3}{2}\right)}, \quad a > 0,\ m > -1,\ n > -2.$

600. $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}, & n > -1, \\ \text{or} & \\ \frac{(n-1)!!}{n!!}\frac{\pi}{2}, & n \neq 0,\ n \text{ is an even integer,} \\ \text{or} & \\ \frac{(n-1)!!}{n!!}, & n \neq 1,\ n \text{ is an odd integer.} \end{cases}$

601. $\int_0^\infty \frac{\sin ax}{x}\,dx = \begin{cases} \frac{\pi}{2}, & a > 0, \\ \text{or} & \\ 0, & a = 0, \\ \text{or} & \\ -\frac{\pi}{2}, & a < 0. \end{cases}$

602. $\int_0^\infty \frac{\cos x}{x}\,dx = \infty.$

603. $\int_0^\infty \frac{\tan x}{x}\,dx = \frac{\pi}{2}.$

604. $\int_0^\infty \frac{\tan ax}{x}\,dx = \frac{\pi}{2}, \quad a > 0.$

605. $\int_0^\pi \sin(nx)\sin(mx)\,dx = \int_0^\pi \cos(nx)\cos(mx)\,dx = 0,$

$n \neq m,\ n$ is an integer, m is an integer.

606. $\int_0^{\pi/n} \sin(nx)\cos(nx)\,dx = \int_0^\pi \sin(nx)\cos(nx)\,dx = 0, \quad n$ is an integer.

607. $\int_0^\pi \sin ax\cos bx\,dx = \begin{cases} \frac{2a}{a^2-b^2}, & a-b \text{ is an odd integer.} \\ \text{or} & \\ 0, & a-b \text{ is an even integer.} \end{cases}$

608. $\int_0^\infty \frac{\sin x \cos ax}{x}\,dx = \begin{cases} 0, & |a| > 1, \\ \text{or} & \\ \frac{\pi}{4}, & |a| = 1, \\ \text{or} & \\ \frac{\pi}{2}, & |a| < 1. \end{cases}$

609. $\int_0^\infty \frac{\sin ax \sin bx}{x^2}\,dx = \begin{cases} \frac{\pi a}{2}, & 0 < a \le b, \\ \text{or} & \\ \frac{\pi b}{2}, & 0 < b \le a. \end{cases}$

610. $\int_0^\pi \sin^2 mx\,dx = \int_0^\pi \cos^2 mx\,dx = \frac{\pi}{2}$, m is an integer.

611. $\int_0^\infty \frac{\sin^2 px}{x^2}\,dx = \frac{\pi\,|p|}{2}.$

612. $\int_0^\infty \frac{\sin x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\sin(p\pi/2)}, \quad 0 < p < 1.$

613. $\int_0^\infty \frac{\cos x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\cos(p\pi/2)}, \quad 0 < p < 1.$

614. $\int_0^\infty \frac{1-\cos px}{x^2}\,dx = \frac{\pi\,|p|}{2}.$

615. $\int_0^\infty \frac{\sin px \cos qx}{x}\,dx = \begin{cases} 0, & q > p > 0, \\ \text{or} & \\ \frac{\pi}{2}, & p > q > 0, \\ \text{or} & \\ \frac{\pi}{4}, & p = q > 0. \end{cases}$

616. $\int_0^\infty \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi}{2\,|a|}e^{-|ma|}.$

617. $\int_0^\infty \cos x^2\,dx = \int_0^\infty \sin x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$

618. $\int_0^\infty \sin(ax^n)\,dx = \frac{1}{na^{1/n}}\Gamma\left(\frac{1}{n}\right)\sin\frac{\pi}{2n}, \quad n > 1.$

619. $\int_0^\infty \cos(ax^n)\,dx = \frac{1}{na^{1/n}}\Gamma\left(\frac{1}{n}\right)\cos\frac{\pi}{2n}, \quad n > 1.$

620. $\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx = \int_0^\infty \frac{\cos x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}.$

621. $\int_0^\infty \frac{\sin^3 x}{x}\,dx = \frac{\pi}{4}.$

622. $\int_0^\infty \frac{\sin^3 x}{x^2}\,dx = \frac{3}{4}\log 3.$

623. $\int_0^\infty \frac{\sin^3 x}{x^3}\,dx = \frac{3\pi}{8}.$

624. $\int_0^\infty \frac{\sin^4 x}{x^4}\,dx = \frac{\pi}{3}.$

625. $\int_0^{\pi/2} \frac{dx}{1+a\cos x}\,dx = \frac{\cos^{-1} a}{\sqrt{1-a^2}}, \quad |a| < 1.$

626. $\int_0^{\pi} \frac{dx}{a + b\cos x}\,dx = \frac{\pi}{\sqrt{a^2 - b^2}}, \quad a > b \geq 0.$

627. $\int_0^{2\pi} \frac{dx}{1 + a\cos x}\,dx = \frac{2\pi}{\sqrt{1 - a^2}}, \quad |a| < 1.$

628. $\int_0^{\infty} \frac{\cos ax - \cos bx}{x}\,dx = \log\left|\frac{b}{a}\right|.$

629. $\int_0^{\pi/2} \frac{dx}{a^2\sin^2 x + b^2\cos^2 x}\,dx = \frac{\pi}{2\,|ab|}.$

630. $\int_0^{\pi/2} \frac{dx}{(a^2\sin^2 x + b^2\cos^2 x)^2}\,dx = \frac{\pi(a^2 + b^2)}{4a^3b^3}, \quad a > 0,\ b > 0.$

631. $\int_0^{\pi/2} \sin^{n-1} x\cos^{m-1} x\,dx = \frac{1}{2}B\left(\frac{n}{2}\right)\frac{m}{2},$

m is a positive integer, n is a positive integer.

632. $\int_0^{\pi/2} \sin^{2n+1} x\,dx = \frac{(2n)!!}{(2n+1)!!}, \quad$ n is a positive integer.

633. $\int_0^{\pi/2} \sin^{2n} x\,dx = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}, \quad$ n is a positive integer.

634. $\int_0^{\pi/2} \frac{x}{\sin x}\,dx = 2\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots\right).$

635. $\int_0^{\pi/2} \frac{dx}{1 + \tan^m x}\,dx = \frac{\pi}{4}, \quad$ m is a non-negative integer.

636. $\int_0^{\pi/2} \sqrt{\cos x}\,dx = \frac{(2\pi)^{3/2}}{(\Gamma(1/4))^2}.$

637. $\int_0^{\pi/2} \tan^h x\,dx = \frac{\pi}{2\cos\left(\frac{h\pi}{2}\right)}, \quad 0 < h < 1.$

638. $\int_0^{\pi/2} \frac{\tan^{-1} ax - \tan^{-1} bx}{x}\,dx = \frac{\pi}{2}\log\frac{a}{b}, \quad a > 0,\ b > 0.$

639. $\int_0^{\infty} e^{-ax}\,dx = \frac{1}{a}, \quad a > 0.$

640. $\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x}\,dx = \log\frac{b}{a}, \quad a > 0,\ b > 0.$

641. $\int_0^{\infty} x^n e^{-ax}\,dx = \begin{cases} \dfrac{\Gamma(n+1)}{a^{n+1}}, & a > 0,\ n > -1, \\ \text{or} & \\ \dfrac{n!}{a^{n+1}}, & a > 0,\ n \text{ is a positive integer.} \end{cases}$

642. $\int_0^{\infty} x^n e^{-ax^p}\,dx = \frac{\Gamma((n+1)/p)}{pa^{(n+1)/p}}, \quad a > 0,\ p > 0,\ n > -1.$

643. $\int_0^{\infty} e^{-a^2x^2}\,dx = \frac{1}{2a}\sqrt{\pi}, \quad a > 0.$

644. $\int_0^{b} e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\operatorname{erf}\left(b\sqrt{a}\right), \quad a > 0.$

645. $\int_b^{\infty} e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\operatorname{erfc}\left(b\sqrt{a}\right), \quad a > 0.$

646. $\int_0^\infty xe^{-x^2}\,dx = \frac{1}{2}.$

647. $\int_0^\infty x^2e^{-x^2}\,dx = \frac{\sqrt{\pi}}{4}.$

648. $\int_0^\infty x^{2n}e^{-ax^2}\,dx = \frac{(2n-1)!!}{2(2a)^n}\sqrt{\frac{\pi}{a}},\quad a>0,\ n>0.$

649. $\int_0^\infty x^{2n+1}e^{-ax^2}\,dx = \frac{n!}{2a^{n+1}},\quad a>0,\ n>-1.$

650. $\int_0^1 x^m e^{-ax}\,dx = \frac{m!}{a^{m+1}}\left[1-e^{-a}\sum_{r=0}^{m}\frac{a^r}{r!}\right].$

651. $\int_0^\infty e^{\left(-x^2-a^2/x^2\right)}\,dx = \frac{e^{-2|a|}\sqrt{\pi}}{2}.$

652. $\int_0^\infty e^{\left(-ax^2-b/x^2\right)}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}},\quad a>0,\ b>0.$

653. $\int_0^\infty \sqrt{x}e^{-ax}\,dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}},\quad a>0.$

654. $\int_0^\infty \frac{e^{-ax}}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{a}},\quad a>0.$

655. $\int_0^\infty e^{-ax}\cos mx\,dx = \frac{a}{a^2+m^2},\quad a>0.$

656. $\int_0^\infty e^{-ax}\cos(bx+c)\,dx = \frac{a\cos c-b\sin c}{a^2+b^2},\quad a>0.$

657. $\int_0^\infty e^{-ax}\sin mx\,dx = \frac{m}{a^2+m^2},\quad a>0.$

658. $\int_0^\infty e^{-ax}\sin(bx+c)\,dx = \frac{b\cos c+a\sin c}{a^2+b^2},\quad a>0.$

659. $\int_0^\infty xe^{-ax}\sin bx\,dx = \frac{2ab}{(a^2+b^2)^2},\quad a>0.$

660. $\int_0^\infty xe^{-ax}\cos bx\,dx = \frac{a^2-b^2}{(a^2+b^2)^2},\quad a>0.$

661. $\int_0^\infty x^n e^{-ax}\sin bx\,dx = \frac{n!\left[(a+ib)^{n+1}-(a-ib)^{n+1}\right]}{2i(a^2+b^2)^{n+1}},\quad a>0.$

662. $\int_0^\infty x^n e^{-ax}\cos bx\,dx = \frac{n!\left[(a-ib)^{n+1}+(a+ib)^{n+1}\right]}{2(a^2+b^2)^{n+1}},\quad a>0,\ n>-1.$

663. $\int_0^\infty \frac{e^{-ax}\sin x}{x}\,dx = \cot^{-1}a,\quad a>0.$

664. $\int_0^\infty e^{-a^2x^2}\cos bx\,dx = \frac{\sqrt{\pi}}{2\,|a|}\exp^{-b^2/(4a^2)},\quad ab>0.$

665. $\int_0^\infty e^{-x\cos\phi}x^{b-1}\sin(x\sin\phi)\,dx = \Gamma(b)\sin(b\phi),\quad b>0,\ -\frac{\pi}{2}<\phi<\frac{\pi}{2}.$

666. $\int_0^\infty e^{-x\cos\phi}x^{b-1}\cos(x\sin\phi)\,dx = \Gamma(b)\cos(b\phi),\quad b>0,\ -\frac{\pi}{2}<\phi<\frac{\pi}{2}.$

667. $\int_0^\infty x^{b-1}\cos x\,dx = \Gamma(b)\cos\left(\frac{b\pi}{2}\right),\quad 0<b<1.$

668. $\int_0^\infty x^{b-1}\sin x\,dx = \Gamma(b)\sin\left(\frac{b\pi}{2}\right), \quad 0<b<1.$

669. $\int_0^1 (\log x)^n\,dx = (-1)^n n!, \quad n>-1.$

670. $\int_0^1 \sqrt{\log\frac{1}{x}}\,dx = \frac{\sqrt{\pi}}{2}.$

671. $\int_0^1 \left(\log\frac{1}{x}\right)^n\,dx = n!.$

672. $\int_0^1 x\log(1-x)\,dx = -\frac{3}{4}.$

673. $\int_0^1 x\log(1+x)\,dx = \frac{1}{4}.$

674. $\int_0^1 x^m(\log x)^n\,dx = \frac{(-1)^n\Gamma(n+1)}{(m+1)^{m+1}}, \quad m>-1,\ n$ is a non-negative integer.

675. $\int_0^1 \frac{\log x}{1+x}\,dx = -\frac{\pi^2}{12}.$

676. $\int_0^1 \frac{\log x}{1-x}\,dx = -\frac{\pi^2}{6}.$

677. $\int_0^1 \frac{\log(1+x)}{x}\,dx = \frac{\pi^2}{12}.$

678. $\int_0^1 \frac{\log(1-x)}{x}\,dx = -\frac{\pi^2}{6}.$

679. $\int_0^1 (\log x)\log(1+x)\,dx = 2-2\log 2-\frac{\pi^2}{12}.$

680. $\int_0^1 (\log x)\log(1-x)\,dx = 2-\frac{\pi^2}{6}.$

681. $\int_0^1 \frac{\log x}{1-x^2}\,dx = -\frac{\pi^2}{8}.$

682. $\int_0^1 \log\left(\frac{1+x}{1-x}\right)\frac{dx}{x} = \frac{\pi^2}{4}.$

683. $\int_0^1 \frac{\log x}{\sqrt{1-x^2}}\,dx = -\frac{\pi}{2}\log 2.$

684. $\int_0^1 x^m\left[\log\left(\frac{1}{x}\right)\right]^n\,dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}, \quad m>-1,\ n>-1.$

685. $\int_0^1 \frac{x^p-x^q}{\log x}\,dx = \log\left(\frac{p+1}{q+1}\right), \quad p>-1,\ q>-1.$

686. $\int_0^1 \frac{dx}{\sqrt{\log(-\log x)}} = \sqrt{\pi}.$

687. $\int_0^\infty \log\left(\frac{e^x+1}{e^x-1}\right)dx = \frac{\pi^2}{4}.$

688. $\int_0^{\pi/2} \log\sin x\,dx = \int_0^{\pi/2}\log\cos x\,dx = -\frac{\pi}{2}\log 2.$

689. $\int_0^{\pi/2} \log\sec x\,dx = \int_0^{\pi/2}\log\operatorname{cosec} x\,dx = \frac{\pi}{2}\log 2.$

690. $\int_0^{\pi} x \log \sin x \, dx = -\frac{\pi^2}{2} \log 2.$

691. $\int_0^{\pi/2} (\sin x) \log \sin x \, dx = \log 2 - 1.$

692. $\int_0^{\pi/2} \log \tan x \, dx = 0.$

693. $\int_0^{\pi} \log (a \pm b \cos x) \, dx = \pi \log \left(\frac{a + \sqrt{a^2 - b^2}}{2} \right), \quad a \geq b.$

694. $\int_0^{\pi} \log (a^2 - 2ab \cos x + b^2) \, dx = \begin{cases} 2\pi \log a, & a \geq b > 0, \\ \quad \text{or} & \\ 2\pi \log b, & b \geq a > 0. \end{cases}$

695. $\int_0^{\infty} \frac{\sin ax}{\sinh bx} \, dx = \frac{\pi}{2b} \tanh \frac{a\pi}{2\,|b|}.$

696. $\int_0^{\infty} \frac{\cos ax}{\cosh bx} \, dx = \frac{\pi}{2b} \operatorname{sech} \frac{a\pi}{2b}.$

697. $\int_0^{\infty} \frac{dx}{\cosh ax} = \frac{\pi}{2\,|a|}.$

698. $\int_0^{\infty} \frac{x}{\sinh ax} \, dx = \frac{\pi^2}{4a^2}, \quad a \geq 0.$

699. $\int_0^{\infty} e^{-ax} \cosh (bx) \, dx = \frac{a}{a^2 - b^2}, \quad |b| < a.$

700. $\int_0^{\infty} e^{-ax} \sinh (bx) \, dx = \frac{b}{a^2 - b^2}, \quad |b| < a.$

701. $\int_0^{\infty} \frac{\sinh ax}{e^{bx} + 1} \, dx = \frac{\pi}{2b} \csc \frac{a\pi}{b} - \frac{1}{2a}, \quad b \geq 0.$

702. $\int_0^{\infty} \frac{\sinh ax}{e^{bx} - 1} \, dx = \frac{1}{2a} - \frac{\pi}{2b} \cot \frac{a\pi}{b}, \quad b \geq 0.$

703. $$\int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{\pi}{2} \left[1 + \left(\frac{1}{2} \right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 k^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 k^6 + \dots \right], \quad k^2 < 1.$$

704. $$\int_0^{\pi/2} \frac{dx}{(1 - k^2 \sin^2 x)^{3/2}} = \frac{\pi}{2} \left[1 + \left(\frac{1}{2} \right)^2 3k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 5k^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 7k^6 + \dots \right], \quad k^2 < 1.$$

705. $$\int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 x} \, dx = \frac{\pi}{2} \left[1 - \left(\frac{1}{2} \right)^2 k^2 - \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \frac{k^4}{3} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 \frac{k^6}{5} - \dots \right], \quad k^2 < 1.$$

706. $\int_0^{\infty} e^{-x} \log x \, dx = -\gamma.$

707. $\int_0^{\infty} e^{-x^2} \log x \, dx = -\frac{\sqrt{\pi}}{4} (\gamma + 2 \log 2).$

708. $\int_0^{\infty} \left(\frac{1}{1 - e^{-x}} - \frac{1}{x} \right) e^{-x} \, dx = \gamma.$

709. $\int_0^\infty \frac{1}{x}\left(\frac{1}{1-e^{-x}} - \frac{1}{x}\right)\,dx = \gamma.$

5.5.1 TABLE OF SEMI-INTEGRALS

	f	$\frac{d^{-1/2}f}{dx^{-1/2}}$
(1)	0	0
(2)	1	$2\sqrt{\frac{x}{\pi}}$
(3)	$x^{-1/2}$	$\sqrt{\pi}$
(4)	x	$\frac{4x^{2/3}}{3\sqrt{\pi}}$
(5)	$x^n,\ n=0,1,2,\ldots$	$\frac{(n!)^2(4x)^{n+1/2}}{(2n+1)!\sqrt{\pi}}$
(6)	$x^p,\ p>-1$	$\frac{\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}x^{p+1/2}$
(7)	$\sqrt{1+x}$	$\sqrt{\frac{x}{\pi}} + \frac{(1+x)\tan^{-1}\left(\sqrt{x}\right)}{\sqrt{\pi}}$
(8)	$\frac{1}{\sqrt{1+x}}$	$\frac{2}{\sqrt{\pi}}\tan^{-1}\left(\sqrt{x}\right)$
(9)	$\frac{1}{1+x}$	$\frac{2\sinh^{-1}\left(\sqrt{x}\right)}{\sqrt{\pi(1+x)}}$
(10)	e^x	$e^x \operatorname{erf}\left(\sqrt{x}\right)$
(11)	$e^x \operatorname{erf}\left(\sqrt{x}\right)$	$e^x - 1$
(12)	$\sin\left(\sqrt{x}\right)$	$\sqrt{\pi x}J_1\left(\sqrt{x}\right)$
(13)	$\cos\left(\sqrt{x}\right)$	$\sqrt{\pi x}H_{-1}\left(\sqrt{x}\right)$
(14)	$\sinh\left(\sqrt{x}\right)$	$\sqrt{\pi x}I_1\left(\sqrt{x}\right)$
(15)	$\cosh\left(\sqrt{x}\right)$	$\sqrt{\pi x}L_{-1}\left(\sqrt{x}\right)$
(16)	$\frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}}$	$\sqrt{\pi}H_0\left(\sqrt{x}\right)$
(17)	$\frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}$	$\sqrt{\pi}J_0\left(\sqrt{x}\right)$
(18)	$\log x$	$2\sqrt{\frac{x}{\pi}}\left[\log(4x) - 2\right]$
(19)	$\frac{\log x}{\sqrt{x}}$	$\sqrt{\pi}\log\left(\frac{x}{4}\right)$