7.3 Kruskal's Algorithm

- **Kruskal's algorithm:** finds an MST of a weighted network; at each step, it chooses the cheapest available edge that does not form a circuit, **216**
- **negative network:** the network obtained by changing the signs of the weights in the original network, **217**
- **Kruskal's algorithm for MaxSTs:** applies Kruskal's algorithm for MSTs to the negative network, **218**

EXERCISES

WALKING

7.1 Networks and Trees

1. A computer lab has seven computers labeled A through G. The connections between computers are as follows:
   - A is connected to D and G
   - B is connected to C, E, and F
   - C is connected to B, E, and F
   - D is connected to A and G
   - E is connected to B and C
   - F is connected to B and C
   - G is connected to A and D

   Is the lab set-up a computer network? Explain why or why not.

2. The following is a list of the electrical power lines connecting eight small towns labeled A through H.
   - A power line connecting A and D
   - A power line connecting B and C
   - A power line connecting B and E
   - A power line connecting B and G
   - A power line connecting C and G
   - A power line connecting D and F
   - A power line connecting D and H
   - A power line connecting E and G

   Do the power lines form a network? Explain why or why not.

3. Consider the network shown in Fig. 7-20.
   (a) How many degrees of separation are there between C and E?
   (b) How many degrees of separation are there between A and E?
   (c) How many degrees of separation are there between A and H?

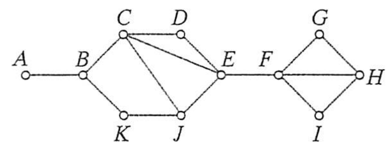

FIGURE 7-20

4. Consider the network shown in Fig. 7-21.
   (a) How many degrees of separation are there between D and J?
   (b) How many degrees of separation are there between A and L?
   (c) How many degrees of separation are there between A and K?

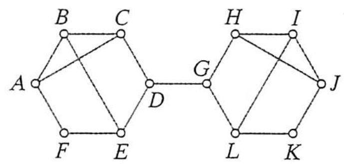

FIGURE 7-21

5. Consider the tree shown in Fig. 7-22 on the next page.
   (a) How many degrees of separation are there between A and J?
   (b) How many degrees of separation are there between E and L?
   (c) How many degrees of separation are there between M and P?
   (d) What is the largest degree of separation between a pair of vertices?

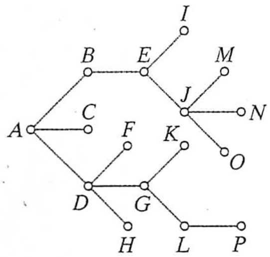

FIGURE 7-22

6. Consider the tree shown in Fig. 7-23.

(a) How many degrees of separation are there between A and P?

(b) How many degrees of separation are there between E and P?

(c) How many degrees of separation are there between L and P?

(d) What is the largest degree of separation between a pair of vertices?

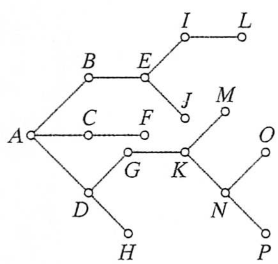

FIGURE 7-23

In Exercises 7 through 20 you are given information about a network. Choose one of the following three options: (A) the network is definitely a tree; (B) the network is definitely not a tree; (C) the network may or may not be a tree (more information is needed). Accompany your answer with a brief explanation for your choice.

7. The network has 15 vertices and 16 edges.

8. The network has 23 vertices and no bridges.

9. The network has 16 vertices and 15 edges.

10. The network has 23 vertices and 22 bridges.

11. The network has redundancy $R = 1$.

12. The network has redundancy $R = 0$.

13. The network has 10 vertices (A through J), and there is only one path connecting A and J.

14. The network has 10 vertices (A through J) and there are two paths connecting C and D.

15. The network has five vertices, no loops, and no multiple edges, and every vertex has degree 4.

16. The network has five vertices, no loops, and no multiple edges, and every vertex has degree 2.

17. The network has five vertices, no loops, and no multiple edges, and has one vertex of degree 4 and four vertices of degree 1.

18. The network has five vertices, no loops, and no multiple edges, and has two vertices of degree 1 and three vertices of degree 2.

19. The network has all vertices of even degree. (*Hint*: You will need to use some concepts from Chapter 5 to answer this question.)

20. The network has two vertices of odd degree and all the other vertices of even degree. (*Hint*: You will need to use some concepts from Chapter 5 to answer this question.)

7.2 Spanning Trees, MSTs, and MaxSTs

21. Consider the network shown in Fig. 7-24.

(a) Find a spanning tree of the network.

(b) Calculate the redundancy of the network.

(c) What is the largest degree of separation between a pair of vertices in the network?

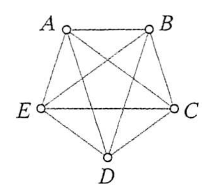

FIGURE 7-24

22. Consider the network shown in Fig. 7-25.

(a) Find a spanning tree of the network.

(b) Calculate the redundancy of the network.

(c) What is the largest degree of separation between a pair of vertices in the network?

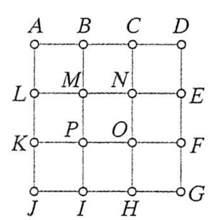

FIGURE 7-25

23. Consider the network shown in Fig. 7-26.

(a) Find a spanning tree of the network.

(b) Calculate the redundancy of the network.

(c) What is the largest degree of separation between a pair of vertices in the network?

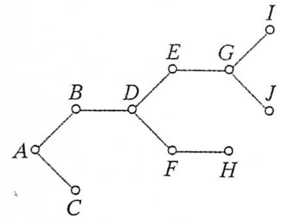

FIGURE 7-26

24. Consider the network shown in Fig. 7-27.

(a) Find a spanning tree of the network.

(b) Calculate the redundancy of the network.

(c) What is the largest degree of separation between a pair of vertices in the network?

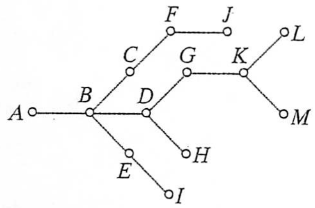

FIGURE 7-27

25. **(a)** Find all the spanning trees of the network shown in Fig. 7-28(a).

(b) Find all the spanning trees of the network shown in Fig. 7-28(b).

(c) How many different spanning trees does the network shown in Fig. 7-28(c) have?

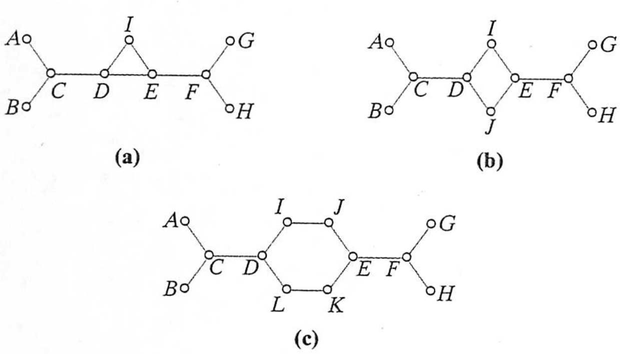

FIGURE 7-28

26. **(a)** Find all the spanning trees of the network shown in Fig. 7-29(a).

(b) Find all the spanning trees of the network shown in Fig. 7-29(b).

(c) How many different spanning trees does the network shown in Fig. 7-29(c) have?

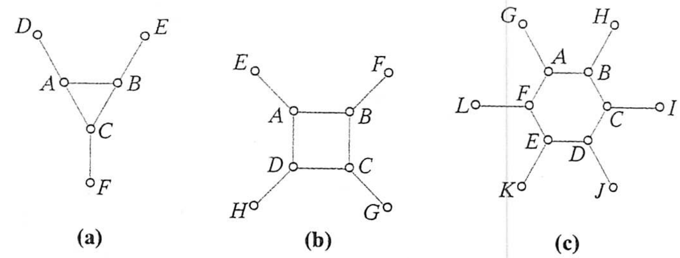

FIGURE 7-29

27. **(a)** How many different spanning trees does the network shown in Fig. 7-30(a) have?

(b) How many different spanning trees does the network shown in Fig. 7-30(b) have?

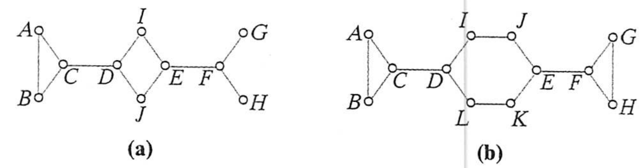

FIGURE 7-30

28. **(a)** How many different spanning trees does the network shown in Fig. 7-31(a) have?

(b) How many different spanning trees does the network shown in Fig. 7-31(b) have?

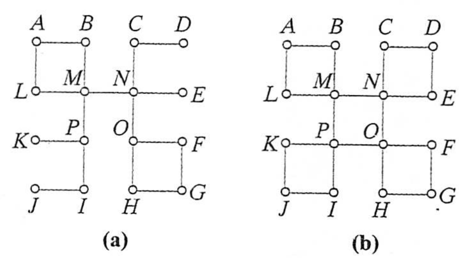

FIGURE 7-31

29. Consider the network shown in Fig. 7-32.

(a) How many different spanning trees does this network have?

(b) Find the spanning tree that has the largest degree of separation between *H* and *G*.

(c) Find a spanning tree that has the smallest degree of separation between *H* and *G*.

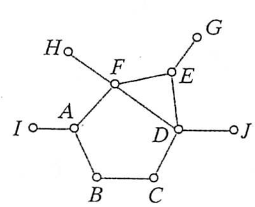

FIGURE 7-32

49. Suppose that in a weighted network there is just one edge (call it XY) with the *smallest* weight. Explain why the edge XY must be in every MST of the network.

50. Suppose that in a weighted network there is just one edge (call it XY) with the *largest* weight.

(a) Give an example of a network with more than one MST and such that XY must be in every MST.

(b) Give an example of a network with more than one MST and such that XY is in none of the MSTs.

51. Suppose G is a disconnected graph with N vertices, M edges, and no circuits.

(a) How many components does the graph have when $N = 9$ and $M = 6$?

(b) How many components does the graph have when $N = 240$ and $M = 236$? Explain your answer.

52. Suppose G is a disconnected graph with no circuits. Let N denote the number of vertices, M the number of edges, and K the number of components. Explain why $M = N - K$. (*Hint*: Try Exercise 51 first.)

53. Cayley's theorem. Cayley's theorem says that the number of spanning trees in a complete graph with N vertices is given by N^{N-2}.

(a) List the $4^2 = 16$ spanning trees of K_4.

(b) Which is larger, the number of Hamilton circuits or the number of spanning trees in a complete graph with N vertices? Explain.

RUNNING

54. Show that if a tree has a vertex of degree K, then there are at least K vertices in the tree of degree 1.

55. A *bipartite graph* is a graph with the property that the vertices of the graph can be divided into two sets A and B so that every edge of the graph joins a vertex from A to a vertex from B (Fig. 7-43). Explain why trees are always bipartite graphs.

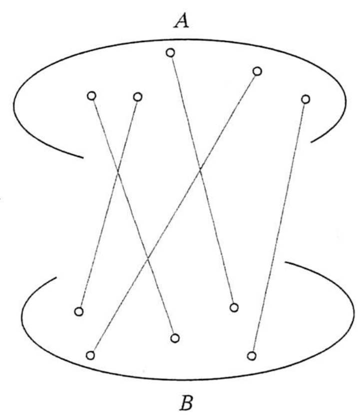

FIGURE 7-43

56. Suppose that there is an edge in a network that must be included in any spanning tree. Give an algorithm for finding the minimum spanning tree that includes a given edge. (*Hint*: Modify Kruskal's algorithm.)

PROJECTS AND PAPERS

1 Other Algorithms for MSTs and MaxSTs

Kruskal's algorithm is not the only good algorithm for finding MSTs and MaxSTs in a weighted network—two other well-known algorithms that are used to solve the same problem are *Prim's algorithm* and *Boruvka's algorithm*. Prepare a presentation on Prim's and Boruvka's algorithms. Describe both algorithms, and demonstrate how they work using one of the examples from the chapter. Compare Prim's and Boruvka's algorithms to Kruskal's algorithm, and discuss their similarities and differences.

2 Dijkstra's Shortest-Path Algorithm

In a network, there is at least one, but often many, paths connecting two vertices X and Y. When the network is an ordinary network, the "shortest path" between X and Y is defined as the path with the fewest number of edges (and the number of edges in a shortest path is called the *degree of separation* between X and Y). On the other hand, when the network is a *weighted* network the "shortest path" between X and Y has a different definition: Among all paths connecting X and Y, it is the path of *least total weight.*

Finding the *shortest path* between pairs of vertices in a weighted network is a problem that has many similarities to the problem of finding a *minimum spanning tree* of the network, and there is a nice algorithm that solves the problem optimally and efficiently. The algorithm, known as *Dijkstra's algorithm*, is named after the Dutch computer scientist Edsger Dijkstra, who first proposed the algorithm in 1959. In this project you are asked to prepare a presentation on Dijkstra's algorithm. At the very least you should carefully describe the algorithm, discuss some of its applications to real-world problems, and illustrate the algorithm with a couple of examples. Dijkstra's algorithm is a well-known algorithm, and you will find plenty of resources for this project on the Web.

3 Social Networks and Privacy

There is an inverse relation between connectedness and privacy. In general, the more connected you are the less privacy you have. The problem gets worse when it comes to