- **Fleury's algorithm:** an algorithm for finding Euler circuits or Euler paths in a graph; it builds the Euler circuit (path) edge by edge—choosing a bridge of the yet-to-be traveled part of the graph only when there is no other choice, **154**

5.4 Eulerizing and Semi-Eulerizing Graphs

- **eulerization:** the process of duplicating edges in a graph to make it have all even vertices, **158**
- **semi-eulerization:** the process of duplicating edges in a graph to make it have all but two even vertices, **161**

EXERCISES

WALKING

5.1 Street-Routing Problems

No exercises for this section.

5.2 An Introduction to Graphs

1. For the graph shown in Fig. 5-29,
   - **(a)** give the vertex set.
   - **(b)** give the edge list.
   - **(c)** give the degree of each vertex.
   - **(d)** draw a version of the graph without crossing points.

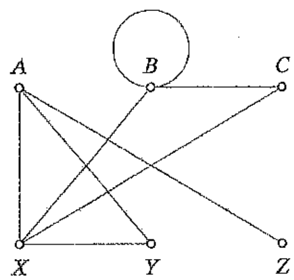


FIGURE 5-29

2. For the graph shown in Fig. 5-30,
   - **(a)** give the vertex set.
   - **(b)** give the edge list.
   - **(c)** give the degree of each vertex.
   - **(d)** sraw a version of the graph without crossing points.

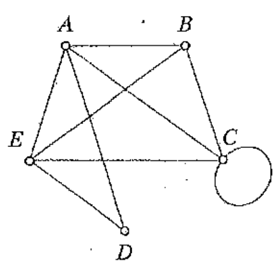


FIGURE 5-30

3. For the graph shown in Fig. 5-31,
   - **(a)** give the vertex set.
   - **(b)** give the edge list.
   - **(c)** give the degree of each vertex.
   - **(d)** give the number of components of the graph.

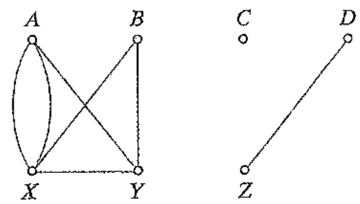


FIGURE 5-31

4. For the graph shown in Fig. 5-32,
   - **(a)** give the vertex set.
   - **(b)** give the edge list.
   - **(c)** give the degree of each vertex.
   - **(d)** give the number of components of the graph.

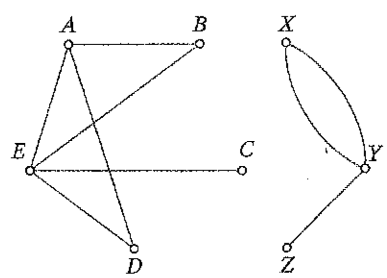


FIGURE 5-32

5. Consider the graph with vertex set $\{K, R, S, T, W\}$ and edge list RS, RT, TT, TS, SW, WW, WS. Draw two different pictures of the graph.

6. Consider the graph with vertex set $\{A, B, C, D, E\}$ and edge list $AC, AE, BD, BE, CA, CD, CE, DE$. Draw two different pictures of the graph.

7. Consider the graph with vertex set $\{A, B, C, D, E\}$ and edge list AD, AE, BC, BD, DD, DE. Without drawing a picture of the graph,
   - **(a)** list all the vertices adjacent to D.
   - **(b)** list all the edges adjacent to BD.
   - **(c)** find the degree of D.
   - **(d)** find the sum of the degrees of the vertices.

8. Consider the graph with vertex set $\{A, B, C, X, Y, Z\}$ and edge list $AX, AY, AZ, BB, CX, CY, CZ, YY$. Without drawing a picture of the graph,
   - **(a)** list all the vertices adjacent to Y.
   - **(b)** list all the edges adjacent to AY.
   - **(c)** find the degree of Y.
   - **(d)** find the sum of the degrees of the vertices.

9. **(a)** Give an example of a connected graph with six vertices such that each vertex has degree 2.
   - **(b)** Give an example of a disconnected graph with six vertices such that each vertex has degree 2.
   - **(c)** Give an example of a graph with six vertices such that each vertex has degree 1.

10. **(a)** Give an example of a connected graph with eight vertices such that each vertex has degree 3.
    - **(b)** Give an example of a disconnected graph with eight vertices such that each vertex has degree 3.
    - **(c)** Give an example of a graph with eight vertices such that each vertex has degree 1.

11. Consider the graph in Fig. 5-33.
    - **(a)** Find a path from C to F passing through vertex B but not through vertex D.
    - **(b)** Find a path from C to F passing through both vertex B and vertex D.
    - **(c)** Find a path of length 4 from C to F.
    - **(d)** Find a path of length 7 from C to F.
    - **(e)** How many paths are there from C to A?
    - **(f)** How many paths are there from H to F?
    - **(g)** How many paths are there from C to F?

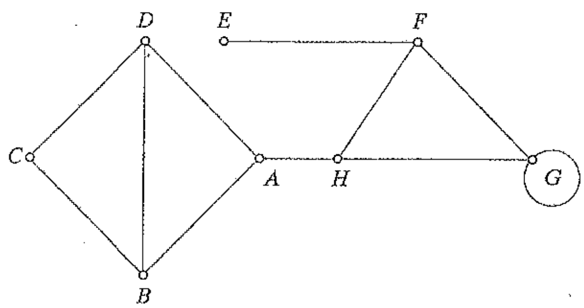


FIGURE 5-33

12. Consider the graph in Fig. 5-33.
    - **(a)** Find a path from D to E passing through vertex G only once.
    - **(b)** Find a path from D to E passing through vertex G twice.
    - **(c)** Find a path of length 4 from D to E.
    - **(d)** Find a path of length 8 from D to E.
    - **(e)** How many paths are there from D to A?
    - **(f)** How many paths are there from H to E?
    - **(g)** How many paths are there from D to E?

13. Consider the graph in Fig. 5-33.
    - **(a)** Find all circuits of length 1.
    - **(b)** Find all circuits of length 2.
    - **(c)** Find all circuits of length 3.
    - **(d)** Find all circuits of length 4.
    - **(e)** What is the total number of circuits in the graph?

14. Consider the graph in Fig. 5-34.
    - **(a)** Find all circuits of length 1.
    - **(b)** Find all circuits of length 2.
    - **(c)** Find all circuits of length 3.
    - **(d)** Find all circuits of length 4.
    - **(e)** Find all circuits of length 5.
    - **(f)** What is the total number of circuits in the graph?

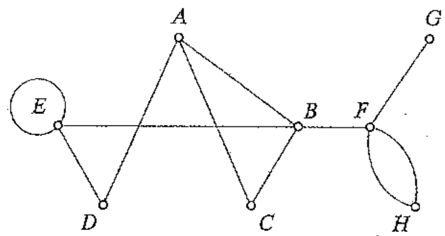


FIGURE 5-34

15. List all the bridges in each of the following graphs:
    - **(a)** the graph in Fig. 5-33.
    - **(b)** the graph with vertex set $\{A, B, C, D, E\}$ and edge list AB, AE, BC, CD, DE.
    - **(c)** the graph with vertex set $\{A, B, C, D, E\}$ and edge list AB, BC, BE, CD.

16. List all the bridges in each of the following graphs:
    - **(a)** the graph in Fig. 5-34.
    - **(b)** the graph with vertex set $\{A, B, C, D, E\}$ and edge list AB, AD, AE, BC, CE, DE.
    - **(c)** the graph with vertex set $\{A, B, C, D, E\}$ and edge list AB, BC, CD, DE.

17. Consider the graph in Fig. 5-35.

(a) Find the largest clique in this graph.

(b) List all the bridges in this graph.

(c) If you remove *all* the bridges from the graph, how many components will the resulting graph have?

(d) Find the *shortest* path (i.e., the path with least length) from C to J. What is the length of the shortest path?

(e) Find a *longest* path from C to J. What is the length of a longest path?

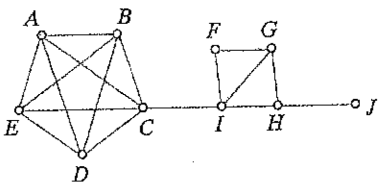


FIGURE 5-35

18. Consider the graph in Fig. 5-36.

(a) Find the largest clique in this graph.

(b) List all the bridges in this graph.

(c) If you remove *all* the bridges from the graph, how many components will the resulting graph have?

(d) Find the *shortest* path (i.e., the path with least length) from E to J. What is the length of the shortest path?

(e) What is the length of a *longest* path from E to J?

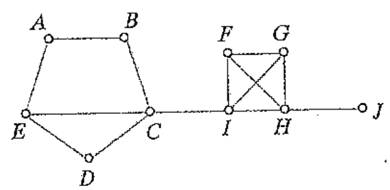


FIGURE 5-36

19. Figure 5-37 shows a map of the downtown area of the picturesque hamlet of Kingsburg, with the Kings River running through the downtown area and the three islands (A, B, and C) connected to each other and both banks by seven bridges. You have been hired by the Kingsburg Chamber of Commerce to organize the annual downtown parade. Part of your job is to plan the route for the parade. Draw a graph that models the layout of Kingsburg.

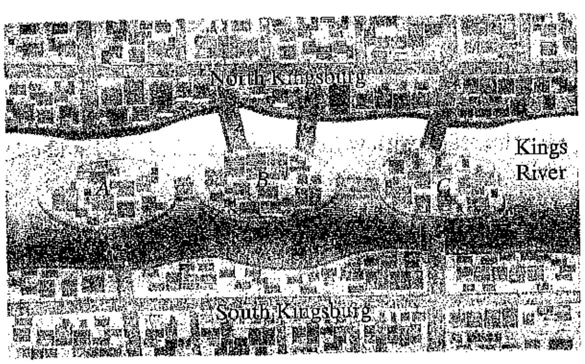


FIGURE 5-37

20. Figure 5-38 is a map of downtown Royalton, showing the Royalton River running through the downtown area and the three islands (A, B, and C) connected to each other and both banks by eight bridges. The Downtown Athletic Club wants to design the route for a marathon through the downtown area. Draw a graph that models the layout of Royalton.

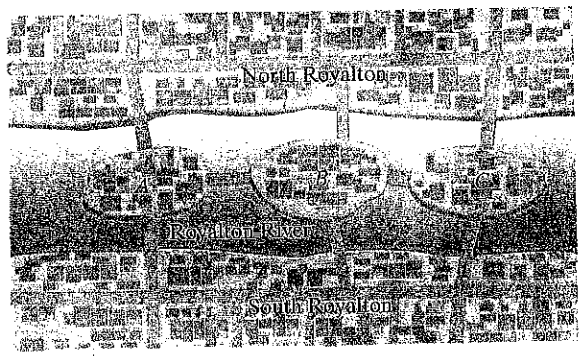


FIGURE 5-38

21. A night watchman must walk the streets of the Green Hills subdivision shown in Fig. 5-39. The night watchman needs to walk only once along each block. Draw a graph that models this street-routing problem.

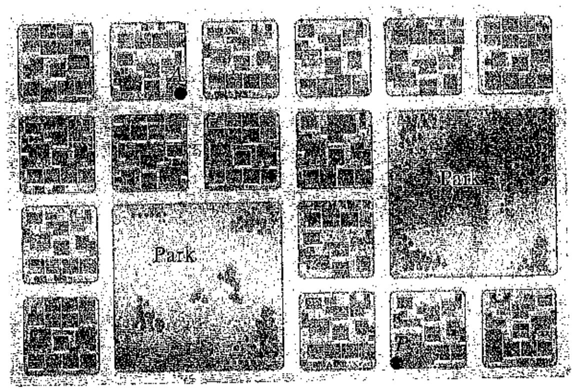


FIGURE 5-39

22. A mail carrier must deliver mail on foot along the streets of the Green Hills subdivision shown in Fig. 5-39. The mail carrier must make two passes on every block that has houses on both sides of the street (once for each side of the street), but only one pass on blocks that have houses on only one side of the street. Draw a graph that models this street-routing problem.

23. Six teams (A, B, C, D, E, and F) are entered in a softball tournament. The top two seeded teams (A and B) have to play only three games; the other teams have to play four games each. The tournament pairings are A plays against C, E, and F; B plays against C, D, and F; C plays against every team except F; D plays against every team except A; E plays against every team except B; and F plays against every team except C. Draw a graph that models the tournament.

24. The Kangaroo Lodge of Madison County has 10 members $(A, B, C, D, E, F, G, H, I, \text{and } J)$. The club has five working committees: the Rules Committee $(A, C, D, E, I, \text{and } J)$, the Public Relations Committee $(B, C, D, H, I, \text{and } J)$, the Guest Speaker Committee $(A, D, E, F, \text{and } H)$, the New Year's Eve Party Committee $(D, F, G, H, \text{and } I)$, and the Fund Raising Committee $(B, D, F, H, \text{and } J)$.

(a) Suppose we are interested in knowing which pairs of members are on the same committee. Draw a graph that models this problem. (*Hint*: Let the vertices of the graph represent the members.)

(b) Suppose we are interested in knowing which committees have members in common. Draw a graph that models this problem. (*Hint*: Let the vertices of the graph represent the committees.)

25. Table 5-3 summarizes the Facebook friendships between a group of eight individuals [an *F* indicates that the individuals (row and column) are Facebook friends]. Draw a graph that models the set of friendships in the group. (Use the first letter of the name to label the vertices.)

	Fred	Pat	Mac	Ben	Tom	Hale	Zac	Cher
Fred		F			F	F		
Pat	F				F	F		F
Mac				F			F	
Ben			F				F	
Tom	F	F				F		
Hale	F	F			F			F
Zac			F	F				
Cher		F				F		

TABLE 5-3

26. The Dean of Students' office wants to know how the seven general education courses selected by incoming freshmen are clustered. For each pair of general education courses, if 30 or more incoming freshmen register for both courses, the courses are defined as being "significantly linked." Table 5-4 shows all the significant links between general education courses (indicated by a 1). Draw a graph that models the significant links between the general education courses. (Use the first letter of each course to label the vertices of the graph.)

	Math	Chemistry	Biology	English	Physics	History	Art
Math		1	1	1	1		
Chemistry	1		1				
Biology	1	1		1		1	
English	1		1		1	1	1
Physics	1			1		1	1
History			1	1	1		1
Art				1	1	1	

TABLE 5-4

27. Figure 5-40 shows the downtown area of the small village of Kenton. The village wants to have a Fourth of July parade that passes through all the blocks of the downtown area, except for the 14 blocks highlighted in yellow, which the police department considers unsafe for the parade route. Draw a graph that models this street-routing problem.

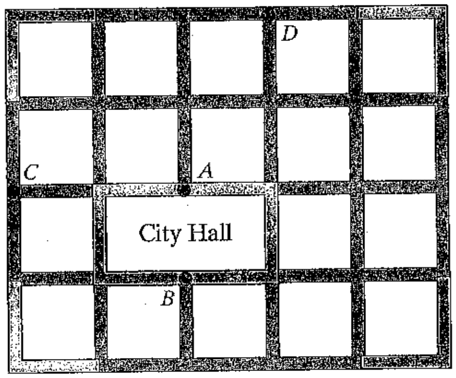


FIGURE 5-40

28. Figure 5-40 shows the downtown area of the small village of Kenton. At regular intervals at night, a police officer must patrol every downtown block at least once, and each of the six blocks along City Hall at least twice. Draw a graph that models this street-routing problem.

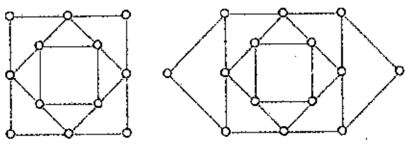
5.3 Euler's Theorems and Fleury's Algorithm

In Exercises 29 through 34 choose from one of the following answers and provide a short explanation for your answer.

(A) *the graph has an Euler circuit.*
(B) *the graph has an Euler path.*
(C) *the graph has neither an Euler circuit nor an Euler path.*
(D) *the graph may or may not have an Euler circuit.*
(E) *the graph may or may not have an Euler path. You do not have to show an actual path or circuit.*

29. **(a)** Fig. 5-41(a) **(b)** Fig. 5-41(b)

(c) A graph with six vertices, all of degree 2

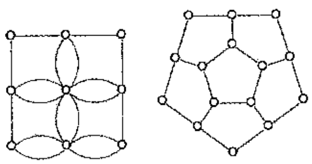


(a) (b)

FIGURE 5-41

30. **(a)** Fig. 5-42(a) **(b)** Fig. 5-42(b)

(c) A graph with eight vertices: six vertices of degree 2 and two vertices of degree 3

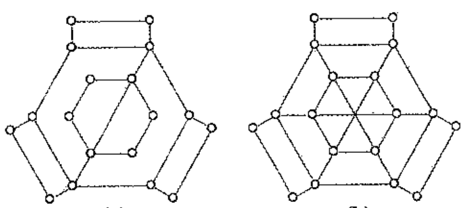


(a) (b)

FIGURE 5-42

31. **(a)** Fig. 5-43(a) **(b)** Fig. 5-43(b)

(c) A disconnected graph with six vertices, all of degree 2

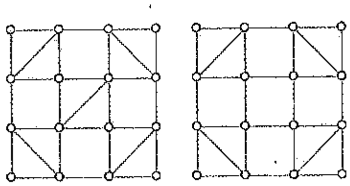


(a) (b)

FIGURE 5-43

32. **(a)** Fig. 5-44(a) **(b)** Fig. 5-44(b)

(c) A disconnected graph with eight vertices: six vertices of degree 2 and two vertices of degree 3

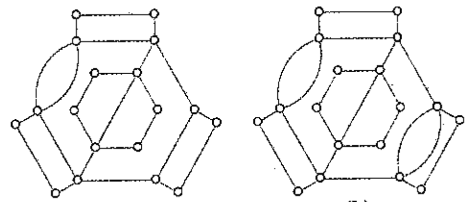


(a) (b)

FIGURE 5-44

33. **(a)** Fig. 5-45(a) **(b)** Fig. 5-45(b)

(c) A graph with six vertices, all of degree 1. [*Hint*: Try Exercise 9(c) first.]

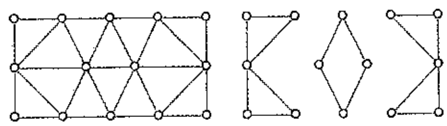


(a) (b)

FIGURE 5-45

34. **(a)** Fig. 5-46(a) **(b)** Fig. 5-46(b)

(c) A graph with eight vertices, all of degree 1.



(a) (b)

FIGURE 5-46

35. Find an Euler circuit for the graph in Fig. 5-47. Show your answer by labeling the edges 1, 2, 3, and so on in the order in which they are traveled.

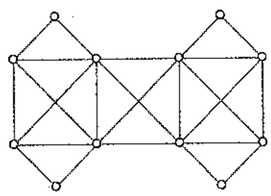


FIGURE 5-47

36. Find an Euler circuit for the graph in Fig. 5-48. Show your answer by labeling the edges 1, 2, 3, and so on in the order in which they can be traveled.

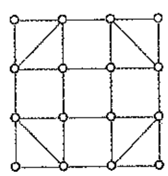


FIGURE 5-48

37. Find an Euler path for the graph in Fig. 5-49. Show your answer by labeling the edges 1, 2, 3, and so on in the order in which they are traveled.

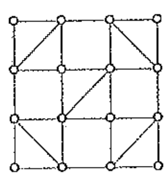


FIGURE 5-49

38. Find an Euler path for the graph in Fig. 5-50. Show your answer by labeling the edges 1, 2, 3, and so on in the order in which they are traveled.

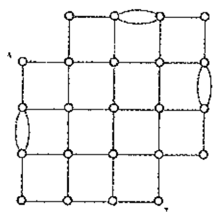


FIGURE 5-50

39. Find an Euler circuit for the graph in Fig. 5-51. Use B as the starting and ending point of the circuit. Show your answer by labeling the edges 1, 2, 3, and so on in the order in which they are traveled.

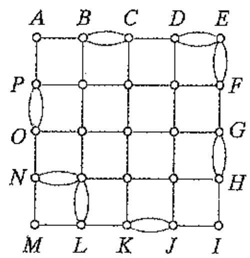


FIGURE 5-51

40. Find an Euler circuit for the graph in Fig. 5-52. Use S as the starting and ending point of the circuit. Show your answer by labeling the edges 1, 2, 3, and so on in the order in which they are traveled.

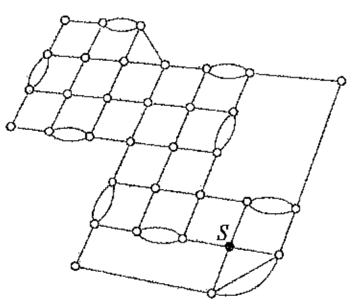


FIGURE 5-52

41. Suppose you are using Fleury's algorithm to find an Euler circuit for a graph and you are in the middle of the process. The graph in Fig. 5-53 shows both the already traveled part of the graph (the red edges) and the yet-to-be traveled part of the graph (the blue edges).

(a) Suppose you are standing at P. What edge(s) could you choose next?

(b) Suppose you are standing at B. What edge should you *not* choose next?

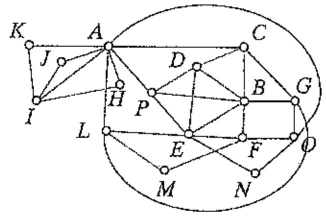


FIGURE 5-53

42. Suppose you are using Fleury's algorithm to find an Euler circuit for a graph and you are in the middle of the process. The graph in Fig. 5-53 shows both the already traveled part of the graph (the red edges) and the yet-to-be traveled part of the graph (the blue edges).

(a) Suppose you are standing at C. What edge(s) could you choose next?

(b) Suppose you are standing at A. What edge should you *not* choose next?

5.4 Eulerizing and Semi-Eulerizating Graphs

43. Find an optimal eulerization for the graph in Fig. 5-54.

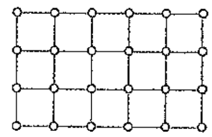


FIGURE 5-54

44. Find an optimal eulerization for the graph in Fig. 5-55.

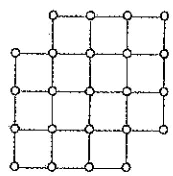


FIGURE 5-55

45. Find an optimal eulerization for the graph in Fig. 5-56.

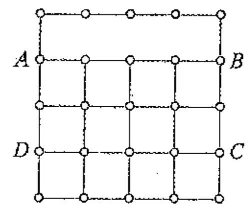


FIGURE 5-56

46. Find an optimal eulerization for the graph in Fig. 5-57.

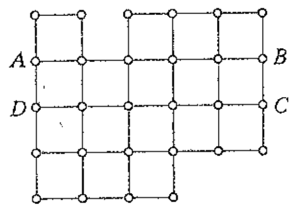


FIGURE 5-57

47. Find an optimal semi-eulerization for the graph in Fig. 5-56. You are free to choose the starting and ending vertices.

48. Find an optimal semi-eulerization for the graph in Fig. 5-57. You are free to choose the starting and ending vertices.

49. Find an optimal semi-eulerization of the graph in Figure 5-56 when A and D are required to be the starting and ending points of the route.

50. Find an optimal semi-eulerization of the graph in Figure 5-57 when A and B are required to be the starting and ending points of the route.

51. Find an optimal semi-eulerization of the graph in Figure 5-56 when B and C are required to be the starting and ending points of the route.

52. Find an optimal semi-eulerization of the graph in Fig. 5-57 when A and D are required to be the starting and ending points of the route.

53. A security guard must patrol on foot the streets of the Green Hills subdivision shown in Fig. 5-39. The security guard wants to start and end his walk at the corner labeled A, and he needs to cover each block of the subdivision at least once. Find an optimal route for the security guard. Describe the route by labeling the edges 1, 2, 3, and so on in the order in which they are traveled. (*Hint*: You should do Exercise 21 first.)

54. A mail carrier must deliver mail on foot along the streets of the Green Hills subdivision shown in Fig. 5-39. His route must start and end at the Post Office, labeled P in the figure. The mail carrier must walk along each block twice if there are houses on both sides of the street and once along blocks where there are houses on only one side of the street. Find an optimal route for the mail carrier. Describe the route by labeling the edges 1, 2, 3, and so on in the order in which they are traveled. (*Hint*: You should do Exercise 22 first.)

55. This exercise refers to the Fourth of July parade problem introduced in Exercise 27. Find an optimal route for the parade that starts at A and ends at B (see Fig. 5-40). Describe the route by labeling the edges 1, 2, 3, . . . etc. in the order they are traveled. [*Hint*: Start with the graph model for the parade route (see Exercise 27); then find an optimal semi-eulerization of the graph that leaves A and B odd; then find an Euler path in this new graph.]

56. This exercise refers to the Fourth of July parade problem introduced in Exercise 27. Find an optimal route for the parade that starts at C and ends at D (see Fig. 5-40). Describe the route by labeling the edges 1, 2, 3, . . . etc. in the order they are traveled. [*Hint*: Start with the graph model for the parade route (see Exercise 27); then find an optimal semi-eulerization of the graph that leaves C and D odd; then find an Euler path in this new graph.]

JOGGING

57. Assume you want to trace the diagram of a basketball court shown in Fig. 5-58 without retracing any lines. How many times would you have to lift your pencil to do it? Explain.

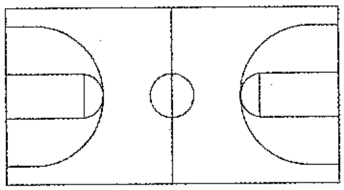


FIGURE 5-58

58. (a) Explain why in every graph the sum of the degrees of all the vertices equals twice the number of edges.

    (b) Explain why every graph must have an even number of odd vertices.

59. If G is a connected graph with no bridges, how many vertices of degree 1 can G have? Explain your answer.

60. **Regular graphs.** A graph is called *regular* if every vertex has the same degree. Let G be a connected regular graph with N vertices.

    (a) Explain why if N is odd, then G must have an Euler circuit.

    (b) When N is even, then G may or may not have an Euler circuit. Give examples of both situations.

61. **Complete bipartite graphs.** A complete bipartite graph is a graph having the property that the vertices of the graph can be divided into two groups A and B and each vertex in A is adjacent to each vertex in B, as shown in Fig. 5-59. Two vertices in A are never adjacent, and neither are two vertices in B. Let m and n denote the number of vertices in A and B, respectively, and assume $m \leq n$.

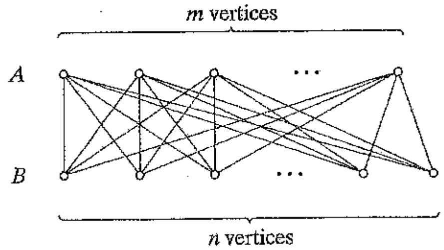


FIGURE 5-59

    (a) Describe all the possible values of m and n for which the complete bipartite graph has an Euler circuit. (*Hint*: There are infinitely many values of m and n.)

(b) Describe all the possible values of m and n for which the complete bipartite graph has an Euler path.

62. Consider the following game. You are given N vertices and are required to build a graph by adding edges connecting these vertices. Each time you add an edge you must pay $1. You can stop when the graph is connected.

(a) Describe the strategy that will cost you the least amount of money.

(b) What is the minimum amount of money needed to build the graph? (Give your answer in terms of N.)

63. Consider the following game. You are given N vertices and allowed to build a graph by adding edges connecting these vertices. For each edge you can add, you make $1. You are not allowed to add loops or multiple edges, and you must stop before the graph is connected (i.e., the graph you end up with must be disconnected).

(a) Describe the strategy that will give you the greatest amount of money.

(b) What is the maximum amount of money you can make building the graph? (Give your answer in terms of N.)

64. Figure 5-60 shows a map of the downtown area of the picturesque hamlet of Kingsburg. You have been hired by the Kingsburg Chamber of Commerce to organize the annual downtown parade. Part of your job is to plan the route for the parade. An *optimal* parade route is one that keeps the bridge crossings to a minimum and yet crosses each of the seven bridges in the downtown area at least once.

(a) Find an optimal parade route if the parade is supposed to start in North Kingsburg but can end anywhere.

(b) Find an optimal parade route if the parade is supposed to start in North Kingsburg and end in South Kingsburg.

(c) Find an optimal parade route if the parade is supposed to start in North Kingsburg and end on island B.

(d) Find an optimal parade route if the parade is supposed to start in North Kingsburg and end on island A.

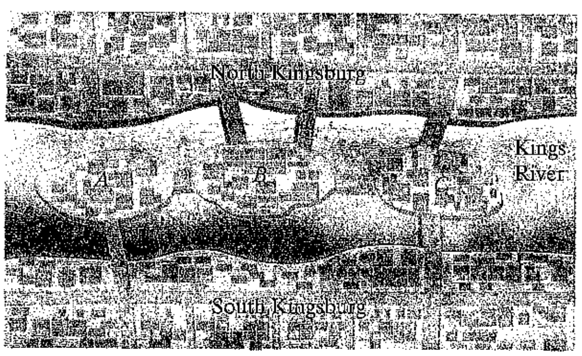


FIGURE 5-60

65. A policeman has to patrol on foot the streets of the subdivision shown in Fig. 5-61. The policeman needs to start his route at the police station, located at X, and end the route at the local coffee shop, located at Y. He needs to cover each block of the subdivision at least once, but he wants to make his route as efficient as possible and duplicate the fewest possible number of blocks.

(a) How many blocks will he have to duplicate in an optimal trip through the subdivision?

(b) Describe an optimal trip through the subdivision. Label the edges 1, 2, 3, and so on in the order the policeman would travel them.

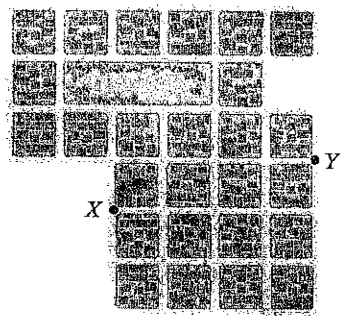


FIGURE 5-61

Exercises 66 through 68 refer to Example 5.23. In this example, the problem is to find an optimal route (i.e., a route with the fewest bridge crossings) for a photographer who needs to cross each of the 11 bridges of Madison County for a photo shoot. The layout of the 11 bridges is shown in Fig. 5-62. You may find it helpful to review Example 5.23 before trying these two exercises.

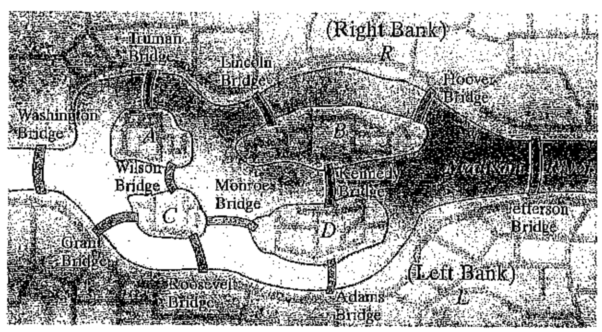


FIGURE 5-62

66. Describe an optimal route for the photographer if the route must start at B and end at L.

67. Describe an optimal route for the photographer if the route must start and end in D and the first bridge crossed must be the Adams Bridge.

68. Describe an optimal route for the photographer if the route must start and end in the same place, the first bridge crossed must be the Adams bridge, and the last bridge crossed must be the Grant Bridge.

69. This exercise comes to you courtesy of Euler himself. Here is the question in Euler's own words, accompanied by the diagram shown in Fig. 5-63.

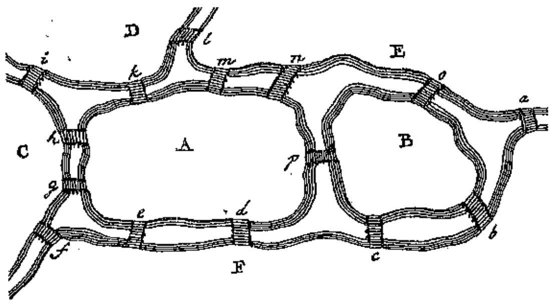


FIGURE 5-63

Let us take an example of two islands with four rivers forming the surrounding water. There are fifteen bridges marked a, b, c, d, etc., across the water around the islands and the adjoining rivers. The question is whether a journey can be arranged that will pass over all the bridges but not over any of them more than once.

What is the answer to Euler's question? If the "journey" is possible, describe it. If it isn't, explain why not.

RUNNING

70. Suppose G is a connected graph with N vertices, all of even degree. Let k denote the number of bridges in G. Find the value(s) of k. Explain your answer.

71. Suppose G is a connected graph with $N-2$ even vertices and two odd vertices. Let k denote the number of bridges in G. Find all the possible values of k. Explain your answer.

72. Suppose G is a disconnected graph with exactly two odd vertices. Explain why the two odd vertices must be in the same component of the graph.

73. Suppose G is a simple graph with N vertices $(N \geq 2)$. Explain why G must have at least two vertices of the same degree.

74. **Kissing circuits.** When two circuits in a graph have no edges in common but share a common vertex v, they are said to be *kissing at* v.

(a) For the graph shown in Fig. 5-64, find a circuit kissing the circuit A, D, C, A (there is only one), and find two different circuits kissing the circuit A, B, D, A.

(b) Suppose G is a connected graph and every vertex in G is even. Explain why the following statement is true: *If a circuit in G has no kissing circuits, then that circuit must be an Euler circuit.*

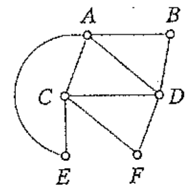


FIGURE 5-64

75. **Hierholzer's algorithm.** *Hierholzer's algorithm* is another algorithm for finding an Euler circuit in a graph. The basic idea behind Hierholzer's algorithm is to start with an arbitrary circuit and then enlarge it by patching to it a *kissing circuit*, continuing this way and making larger and larger circuits until the circuit cannot be enlarged any farther. (For the definition of kissing circuits, see Exercise 74.) More formally, Hierholzer's algorithm is as follows:

Step 1. Start with an arbitrary circuit C_0.

Step 2. Find a kissing circuit to C_0. If there are no kissing circuits to C_0, then you are finished—C_0 is itself an Euler circuit of the graph [see Exercise 74(b)]. If there is a kissing circuit to C_0, let's call it K_0, and let V denote the vertex at which the two circuits kiss. Go to Step 3.

Step 3. Let C_1 denote the circuit obtained by "patching" K_0 to C_0 at vertex V (i.e., start at V, travel along C_0 back to V, and then travel along K_0 back again to V). Now find a kissing circuit to C_1. (If there are no kissing circuits to C_1, then you are finished—C_1 is your Euler circuit.) If there is a kissing circuit to C_1, let's call it K_1, and let W denote the vertex at which the two circuits kiss. Go to Step 4.

Steps 4, 5, and so on. Continue this way until there are no more kissing circuits available.

(a) Use Hierholzer's algorithm to find an Euler circuit for the graph shown in Fig. 5-65 (this is the graph model for the mail carrier in Example 5.14).

(b) Describe a modification of Hierholzer's algorithm that allows you to find an Euler path in a connected graph having exactly two vertices of odd degree. (*Hint:* A path can also have a kissing circuit.)

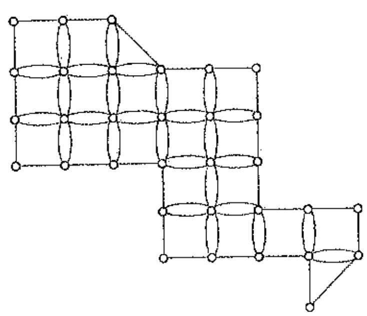


FIGURE 5-65