4.4 Adams's and Webster's Methods

- **Adams's method:** a divisor method that produces an apportionment by using modified quotas that are always rounded up, **115**
- **Webster's method:** a divisor method that produces an apportionment by using modified quotas that are rounded using conventional rounding, **116**

4.5 The Huntington-Hill Method

- **geometric mean:** for two positive numbers, the square root of their product (i.e., geometric mean of a and $b = \sqrt{ab}$.), **118**
- **Huntington-Hill rounding rule:** given a quota q with lower quota L and upper quota U, q is rounded down to L if $q < c$, and rounded up to U if $q > c$, **118**
- **Huntington-Hill method:** a divisor method that produces an apportionment by using modified quotas that are rounded according to the Huntington-Hill rounding rule, **119**

4.6 The Quota Rule and Apportionment Paradoxes

- **quota rule:** a state should never be apportioned less than its lower quota or more than its upper quota, **121**
- **upper-quota violation:** an apportionment of seats to a state that is more than the state's upper quota, **121**
- **lower-quota violation:** an apportionment of seats to a state that is less than the state's lower quota, **121**
- **Alabama paradox:** an apportionment paradox where a state may lose seats to another state merely because of an increase in the number of seats being apportioned, **122**
- **population paradox:** an apportionment paradox where a state may lose seats to another state merely because its population increased at a higher rate, **122**
- **new-states paradox:** an apportionment paradox where a state may lose seats to another state merely because a new state, together with its apportionment of seats, have been added to the apportionment calculations, **124**
- **Balinski and Young's Impossibility Theorem:** a perfect apportionment method (no violations of the quota rule and no apportionment paradoxes) is a mathematical impossibility, **124**

EXERCISES

WALKING

4.1 Apportionment Problems and Apportionment Methods

1. The Bandana Republic is a small country consisting of four states: Apure (population 3,310,000), Barinas (population 2,670,000), Carabobo (population 1,330,000), and Dolores (population 690,000). Suppose that there are $M = 160$ seats in the Bandana Congress, to be apportioned among the four states based on their respective populations.

   (a) Find the standard divisor.

   (b) Find each state's standard quota.

2. The Republic of Wadiya is a small country consisting of four provinces: A (population 4,360,000), B (population 2,280,000), C (population 729,000), and D (population 2,631,000). Suppose that there are $M = 200$ seats in the Wadiya Congress, to be apportioned among the four provinces based on their respective populations.

   (a) Find the standard divisor.

   (b) Find each province's standard quota.

3. The Scotia Metropolitan Area Rapid Transit Service (SMARTS) operates six bus routes (A, B, C, D, E, and F) and 130 buses. The number of buses apportioned to each route is based on the number of passengers riding that route. Table 4-23 shows the daily average ridership (in thousands) on each route.

   (a) Find the standard divisor.

   (b) Explain what the standard divisor represents in this problem.

   (c) Find the standard quotas.

Route	A	B	C	D	E	F
Ridership	45.3	31.07	20.49	14.16	10.26	8.72

TABLE 4-23

4. The Placerville General Hospital has a nursing staff of 225 nurses working in four shifts: A (7:00 A.M. to 1:00 P.M.), B (1:00 P.M. to 7:00 P.M.), C (7:00 P.M. to 1:00 A.M.), and D (1:00 A.M. to 7:00 A.M.). The number of nurses apportioned to each shift is based on the average number of patients treated in that shift, shown in Table 4-24.

   (a) Find the standard divisor.

   (b) Explain what the standard divisor represents in this problem.

   (c) Find the standard quotas.

Shift	A	B	C	D
Patients	871	1029	610	190

TABLE 4-24

5. The Republic of Tropicana is a small country consisting of five states (A, B, C, D, and E). The total population of Tropicana is 27.4 million. According to the Tropicana constitution, the seats in the legislature are apportioned to the states according to their populations. Table 4-25 shows each state's standard quota:

   (a) Find the number of seats in the Tropicana legislature.

   (b) Find the standard divisor.

   (c) Find the population of each state.

State	A	B	C	D	E
Standard quota	41.2	31.9	24.8	22.6	16.5

TABLE 4-25

6. Tasmania State University is made up of five different schools: Agriculture, Business, Education, Humanities, and Science (A, B, E, H, and S for short). The total number of students at TSU is 12,500. The faculty positions at TSU are apportioned to the various schools based on the schools' respective enrollments. Table 4-26 shows each school's standard quota:

   (a) Find the number of faculty positions at TSU.

   (b) Find the standard divisor.

   (c) Find the number of students enrolled in each school.

School	A	B	E	H	S
Standard quota	32.92	15.24	41.62	21.32	138.90

TABLE 4-26

7. According to the 2010 U.S. Census, 8.14% of the U.S. population lived in Texas. Compute Texas's standard quota in 2010 (rounded to two decimal places). (*Hint*: There are 435 seats in the U.S. House of Representatives.)

8. At the time of the 2010 U.S. Census, New York State had a standard quota of 27.3. Estimate what percent of the U.S. population lived in New York State in 2010. Give your answer to the nearest tenth of a percent. (*Hint*: There are 435 seats in the U.S. House of Representatives.)

9. The Interplanetary Federation of Fraternia consists of six planets: Alpha Kappa, Beta Theta, Chi Omega, Delta Gamma, Epsilon Tau, and Phi Sigma (A, B, C, D, E, and F for short). The federation is governed by the Inter-Fraternia Congress, consisting of 200 seats apportioned among the planets according to their populations. Table 4-27 gives the planet populations as percentages of the total population of Fraternia:

   (a) Find the standard divisor (expressed as a percent of the total population).

   (b) Find the standard quota for each planet.

Planet	A	B	C	D	E	F
Population percentage	11.37	8.07	38.62	14.98	10.42	16.54

TABLE 4-27

10. The small island nation of Margarita is made up of four islands: Aleta, Bonita, Corona, and Doritos (A, B, C, and D for short). There are 125 seats in the Margarita Congress, which are apportioned among the islands according to their populations. Table 4-28 gives the island populations as percentages of the total population of Margarita:

(a) Find the standard divisor (expressed as a percent of the total population).

(b) Find the standard quota for each island.

Island	A	B	C	D
Population percentage	6.24	26.16	28.48	39.12

■ TABLE 4-28

4.2 Hamilton's Method

11. Find the apportionment of the Bandana Republic Congress described in Exercise 1 under Hamilton's method.

12. Find the apportionment of the Wadiya Congress described in Exercise 2 under Hamilton's method.

13. Find the apportionment of the SMARTS buses described in Exercise 3 under Hamilton's method.

14. Find the apportionment of the Placerville General Hospital nurses described in Exercise 4 under Hamilton's method.

15. Find the apportionment of the Republic of Tropicana legislature described in Exercise 5 under Hamilton's method.

16. Find the apportionment of the faculty at Tasmania State University described in Exercise 6 under Hamilton's method.

17. Find the apportionment of the Inter-Fraternia Congress described in Exercise 9 under Hamilton's method.

18. Find the apportionment of the Margarita Congress described in Exercise 10 under Hamilton's method.

19. Happy Rivers County consists of three towns: Dunes, Smithville, and Johnstown. Each year the social workers employed by the county are apportioned among the three towns based on the number of cases in each town over the previous calendar year. The number of cases in each town in 2011 is shown in Table 4-29.

(a) Suppose the total number of social workers employed by the county is $M = 24$. Use Hamilton's method to apportion the social workers to the towns based on the caseloads shown in Table 4-29.

(b) Suppose the total number of social workers employed by the county is $M = 25$. Use Hamilton's method to apportion the social workers to the towns based on the caseloads shown in Table 4-29.

(c) Compare your answers in (a) and (b). What is strange about the two apportionments?

Town	Dunes	Smithville	Johnstown
Number of cases	41	106	253

■ TABLE 4-29

20. Plainville Hospital has three wings (A, B, and C). The nurses in the hospital are assigned to the three wings based on the number of beds in each wing, shown in Table 4-30.

(a) Suppose the total number of nurses working at the hospital is $M = 20$. Use Hamilton's method to apportion the nurses to the wings based on Table 4-30.

(b) Suppose an additional nurse is hired at the hospital, bringing the total number of nurses to $M = 21$. Use Hamilton's method to apportion the nurses based on Table 4-30.

(c) Compare your answers in (a) and (b). What is strange about the two apportionments?

Wing	A	B	C
Number of beds	154	66	30

■ TABLE 4-30

4.3 Jefferson's Method

21. The small nation of Fireland is divided into four counties: Arcadia, Belarmine, Crowley, and Dandia. Fireland uses Jefferson's method to apportion the 100 seats in the Chamber of Deputies among the four counties. Table 4-31 shows the populations of the four counties after the most recent census.

(a) Find the standard divisor and the standard quotas for each county.

(b) Determine how many seats would be apportioned if each county was given its lower quota.

(c) Determine how many seats would be apportioned if the divisor $d = 197{,}000$ is used to compute the modified quotas and then all of them are rounded down.

(d) Determine how many seats would be apportioned if the divisor $d = 195{,}000$ is used to compute the modified quotas and then all of them are rounded down.

(e) Determine how many seats would be apportioned if the divisor $d = 195{,}800$ is used to compute the modified quotas and then all of them are rounded down.

(f) Determine how many seats would be apportioned if the divisor $d = 196{,}000$ is used to compute the modified quotas and then all of them are rounded down.

(g) Without doing any additional computations, find three different divisors that would work under Jefferson's method.

County	Arcadia	Belarmine	Crowley	Dandia
Population	4,500,000	4,900,000	3,900,000	6,700,000

■ TABLE 4-31

22. The Republic of Galatia is divided into four provinces: Anline, Brock, Clanwin, and Drundell. Galatia uses Jefferson's method to apportion the 50 seats in its House of Representatives among the four provinces. Table 4-32 shows the populations of the four provinces (in millions) after the most recent census.

(a) Find the standard divisor and the standard quotas for each province.

(b) Determine how many seats would be apportioned if each province was given its lower quota.

(c) Determine how many seats would be apportioned if the divisor $d = 500{,}000$ is used to compute the modified quotas and then all of them are rounded down.

(d) Determine how many seats would be apportioned if the divisor $d = 530{,}000$ is used to compute the modified quotas and then all of them are rounded down.

(e) Determine how many seats would be apportioned if the divisor $d = 520{,}000$ is used to compute the modified quotas and then all of them are rounded down.

(f) Determine how many seats would be apportioned if the divisor $d = 510{,}000$ is used to compute the modified quotas and then all of them are rounded down.

(g) Without doing any additional computations, find three different divisors that would work under Jefferson's method.

Province	Anline	Brock	Clanwin	Drundell
Population (in millions)	5.9	7.8	6.1	6.9

■ TABLE 4-32

23. Find the apportionment of the Bandana Republic Congress as described in Exercise 1 under Jefferson's method.

24. Find the apportionment of the Wadiya Congress described in Exercise 2 under Jefferson's method.

25. Find the apportionment of the SMARTS buses described in Exercise 3 under Jefferson's method.

26. Find the apportionment of the Placerville General Hospital nurses described in Exercise 4 under Jefferson's method.

27. Find the apportionment of the Republic of Tropicana legislature described in Exercise 5 under Jefferson's method.

28. Find the apportionment of the faculty at Tasmania State University described in Exercise 6 under Jefferson's method.

29. Find the apportionment of the Inter-Fraternia Congress described in Exercise 9 under Jefferson's method. (*Hint*: Express the modified divisors in terms of percents of the total population.)

30. Find the apportionment of the Margarita Congress described in Exercise 10 under Jefferson's method. (*Hint*: Express the modified divisors in terms of percents of the total population.)

4.4 Adams's and Webster's Methods

31. Find the apportionment of the Republic of Tropicana legislature described in Exercise 5 under Adams's method.

32. Find the apportionment of the faculty at Tasmania State University described in Exercise 6 under Adams's method.

33. Find the apportionment of the Inter-Fraternia Congress described in Exercise 9 under Adams's method. (*Hint*: Express the modified divisors in terms of percents of the total population.)

34. Find the apportionment of the Margarita Congress described in Exercise 10 under Adams's method. (*Hint*: Express the modified divisors in terms of percents of the total population.)

35. Find the apportionment of the Bandana Republic Congress described in Exercise 1 under Webster's method.

36. Find the apportionment of the Wadiya Congress described in Exercise 2 under Webster's method.

37. Find the apportionment of the Republic of Tropicana legislature discussed in Exercise 5 under Webster's method.

38. Find the apportionment of the faculty at Tasmania State University discussed in Exercise 6 under Webster's method.

39. Find the apportionment of the Inter-Fraternia Congress described in Exercise 9 under Webster's method (*Hint*: Express the modified divisors in terms of percents of the total population.)

40. The small republic of Guayuru (see Example 4.11) consists of five states (*A*, *B*, *C*, *D*, and *E* for short). The populations of the five states are shown in Table 4-33. Find the apportionment of the $M = 40$ seats in the Guayuru House of Representatives under Webster's method.

State	*A*	*B*	*C*	*D*	*E*
Population	34,800	104,800	64,800	140,800	54,800

■ TABLE 4-33

4.5 The Huntington-Hill Method

41. Round each number using the Huntington-Hill rounding rules. (*Hint*: $\sqrt{2} \approx 1.414$ is all the information you need. You won't need a calculator or to look up any tables.)

(a) 1.5

(b) 1.4

(c) 1.41

(d) 1.42

(e) 1.485313

42. Round each number using the Huntington-Hill rounding rules. (*Hint*: $\sqrt{72} \approx 8.48528$ is all the information you need. You won't need a calculator or to look up any tables.)

(a) 8.5

(b) 8.4

(c) 8.483

(d) 8.486

43. In the 2010 apportionment of the U.S. House of Representatives, Rhode Island had a standard quota of 1.488879 and a modified quota of 1.485313. How many seats were apportioned to Rhode Island?

44. In the 2010 apportionment of the U.S. House of Representatives, Missouri had a standard quota of 8.458641 and a modified quota of 8.483. How many seats were apportioned to Missouri?

45. A small country consists of five states: *A*, *B*, *C*, *D*, and *E*. The standard quotas for each state are given in Table 4.34.

(a) Find the number of seats being apportioned.

(b) Find the apportionment under the Huntington-Hill method.

State	*A*	*B*	*C*	*D*	*E*
Standard quota	3.52	10.48	1.41	12.51	12.08

■ TABLE 4-34

46. A small country consists of five states: *A*, *B*, *C*, *D*, and *E*. The standard quotas for each state are given in Table 4.35.

(a) Find the number of seats being apportioned.

(b) Find the apportionment under the Huntington-Hill method.

State	*A*	*B*	*C*	*D*	*E*
Standard quota	25.49	14.52	8.48	30.71	20.8

■ TABLE 4-35

47. A small country consists of five states: *A*, *B*, *C*, *D*, and *E*. The standard quotas for each state are given in Table 4.36.

(a) Find the number of seats being apportioned.

(b) Find the apportionment under the Huntington-Hill method.

State	*A*	*B*	*C*	*D*	*E*
Standard quota	3.46	10.49	1.42	12.45	12.18

■ TABLE 4-36

48. A small country consists of five states: *A*, *B*, *C*, *D*, and *E*. The standard quotas for each state are given in Table 4.37.

(a) Find the number of seats being apportioned.

(b) Find the apportionment under the Huntington-Hill method.

State	*A*	*B*	*C*	*D*	*E*
Standard quota	25.496	14.491	8.486	30.449	21.078

■ TABLE 4-37

49. A country consists of six states, with the state's populations given in Table 4-38. The number of seats to be apportioned is $M = 200$.

(a) Find the apportionment under Webster's method.

(b) Find the apportionment under the Huntington-Hill method.

(c) Compare the apportionments found in (a) and (b).

State	*A*	*B*	*C*	*D*	*E*	*F*
Population	344,970	408,700	219,200	587,210	154,920	285,000

■ TABLE 4-38

50. A country consists of six states, with the state's populations given in Table 4-39. The number of seats to be apportioned is $M = 200$.

(a) Find the apportionment under Webster's method.

(b) Find the apportionment under the Huntington-Hill method.

(c) Compare the apportionments found in (a) and (b).

State	*A*	*B*	*C*	*D*	*E*	*F*
Population	344,970	204,950	515,100	84,860	154,960	695,160

■ TABLE 4-39

4.6 The Quota Rule and Apportionment Paradoxes

51. Suppose you are taking a multiple-choice quiz on the apportionment chapter. The question is to find how many seats are apportioned to state *X* under Hamilton's method. You are given all the necessary information to do the computation, including the fact that the standard quota for state *X* is 35.41. You are presented with four choices: (A) 37, (B) 34, (C) 33, and (D) 36. You can answer this question without doing any work. Which is the correct answer and why?

52. Suppose you are taking a multiple-choice quiz on the apportionment chapter. The question is to find how many seats are apportioned to state *Y* under Hamilton's method. You are given all the necessary information to do the computation, including the fact that the standard quota for state *Y* is 78.24. You are presented with four choices: (A) 80, (B) 77, (C) 79, and (D) 81. You can answer this question without doing any work. Which is the correct answer and why?

53. At the time of the 2000 Census, California's standard quota was 52.45. Under Jefferson's method, California would get an apportionment of 55 seats. What does this say about Jefferson's method?

54. At the time of the 2000 Census, California's standard quota was 52.45. Under Adams's method, California would get an apportionment of 50 seats. What does this say about Adams's method?

55. Suppose you are taking a multiple-choice quiz on the apportionment chapter. The question is to find how many seats are apportioned to state *X* under Jefferson's method. You are given all the necessary information to do the computation,

including the fact that the standard quota for state X is 35.41. You are presented with four choices: (A) 37, (B) 32, (C) 33, and (D) 34. You can answer this question without doing any work. Which is the correct answer and why?

56. Suppose you are taking a multiple-choice quiz on the apportionment chapter. The question is to find how many seats are apportioned to state Y under Adams's method. You are given all the necessary information to do the computation, including the fact that the standard quota for state Y is 78.24. You are presented with four choices: (A) 80, (B) 81, (C) 76, and (D) 82. You can answer this question without doing any work. Which is the correct answer and why?

57. Suppose you are taking a multiple-choice quiz on the apportionment chapter. You are told that the standard quota for state X is 35.41. The question is to determine which of the given answers *can be ruled out as a possible apportionment* to state X under Webster's method. You are presented with five choices: (A) 33, (B) 34, (C) 37, (D) 38, and (E) None of the previous choices can be ruled out. Which is the correct answer and why?

58. Suppose you are taking a multiple-choice quiz on the apportionment chapter. You are told that the standard quota for state Y is 78.24. The question is to determine which of the given answers *can be ruled out as a possible apportionment* to state Y under Webster's method. You are presented with five choices: (A) 77, (B) 76, (C) 80, (D) 81, and (E) None of the previous choices can be ruled out. Which is the correct answer and why?

59. This exercise refers to the apportionment of social workers in Happy Rivers County introduced in Exercise 19. The answers to parts (a) and (b) in Exercise 19 are an illustration of which paradox? Explain. [Obviously, you need to work out parts (a) and (b) of Exercise 19 if you haven't done so yet.]

60. This exercise refers to the apportionment of nurses to wings in Plainsville Hospital introduced in Exercise 20. The answers to parts (a) and (b) in Exercise 20 are an illustration of which paradox? [Obviously, you need to work out parts (a) and (b) of Exercise 20 if you haven't done so yet.]

Exercises 61 and 62 are based on the following story: Mom found an open box of her children's favorite candy bars. She decides to apportion the candy bars among her three youngest children according to the number of minutes each child spent doing homework during the week.

61. **(a)** Suppose that there were 11 candy bars in the box. Given that Bob did homework for a total of 54 minutes, Peter did homework for a total of 243 minutes, and Ron did homework for a total of 703 minutes, apportion the 11 candy bars among the children using Hamilton's method.

(b) Suppose that before mom hands out the candy bars, the children decide to spend a "little" extra time on homework. Bob puts in an extra 2 minutes (for a total of 56 minutes), Peter an extra 12 minutes (for a total of 255 minutes), and Ron an extra 86 minutes (for a total of 789 minutes). Using these new totals, apportion the 11 candy bars among the children using Hamilton's method.

(c) The results of (a) and (b) illustrate one of the paradoxes of Hamilton's method. Which one? Explain.

62. **(a)** Suppose that there were 10 candy bars in the box. Given that Bob did homework for a total of 54 minutes, Peter did homework for a total of 243 minutes, and Ron did homework for a total of 703 minutes, apportion the 10 candy bars among the children using Hamilton's method.

(b) Suppose that just before she hands out the candy bars, mom finds one extra candy bar. Using the same total minutes as in (a), apportion now the 11 candy bars among the children using Hamilton's method.

(c) The results of (a) and (b) illustrate one of the paradoxes of Hamilton's method. Which one? Explain.

63. This exercise comes in two parts. Read Part I and answer (a) and (b), then read Part II and answer (c) and (d).

Part I. The Intergalactic Federation consists of three sovereign planets: Aila, with a population of 5.2 million, Balin, with a population of 15.1 million, and Cona, with a population of 10.6 million. The Intergalactic Parliament has 50 seats that are apportioned among the three planets based on their populations.

(a) Find the standard divisor in the Intergalactic Parliament.

(b) Find the apportionment of the 50 seats to the three planets under Hamilton's method.

Part II. Based on the results of a referendum, the federation expands to include a fourth planet, Dent, with a population of 9.5 million. To account for the additional population the number of seats in the Intergalactic Parliament is increased by 15 to a total of 65. [9.5 million individuals represent approximately 15 seats based on the standard divisor found in (a).]

(c) Find the apportionment of the 65 seats to the four planets using Hamilton's method.

(d) Which paradox is illustrated by the results of (b) and (c)? Explain.

64. This exercise comes in two parts. Read Part I and answer (a) and (b), then read Part II and answer (c) and (d).

Part I. A catering company contracts to provide catering services to three schools: Alexdale, with 617 students, Bromville, with 1,292 students, and Canley, with 981 students. The 30 food-service workers employed by the catering company are apportioned among the schools based on student enrollments.

(a) Find the standard divisor, rounded to the nearest integer.

(b) Find the apportionment of the 30 workers to the three schools under Hamilton's method.

Part II. The catering company gets a contract to service one additional school—Dillwood, with 885 students. To account for the additional students, the company hires 9 additional food-service workers. [885 students represent approximately 9 workers based on the standard divisor found in (a).]

(c) Find the apportionment of the 39 workers to the four schools under Hamilton's method.

(d) Which paradox is illustrated by the results of (b) and (c)? Explain.

JOGGING

65. The small island nation of Margarita consists of four islands: Aleta, Bonita, Corona, and Doritos. The state's population of each island is given in Table 4-40. The number of seats to be apportioned is $M = 100$.

(a) Find the apportionment under the Huntington-Hill method.

(b) Describe any possible quota rule violations that occurred under the apportionment in (a).

State	A	B	C	D
Population	86,915	4,325	5,400	3,360

TABLE 4-40

66. Consider an apportionment problem with N states. The populations of the states are given by $p_1, p_2, \ldots, p_N$, and the standard quotas are $q_1, q_2, \ldots, q_N$, respectively. Describe in words what each of the following quantities represents.

(a) $q_1 + q_2 + \cdots + q_N$

(b) $\dfrac{p_1 + p_2 + \cdots + p_N}{q_1 + q_2 + \cdots + q_N}$

(c) $\left(\dfrac{p_N}{p_1 + p_2 + \cdots + p_N}\right) \times 100$

67. For an arbitrary state X, let q represent its standard quota and s represent the number of seats apportioned to X under some unspecified apportionment method. Interpret in words the meaning of each of the following mathematical statements.

(a) $s - q \geq 1$

(b) $q - s \geq 1$

(c) $|s - q| \leq 0.5$

(d) $0.5 < |s - q| < 1$

68. Consider the problem of apportioning $M = 3$ seats between two states, A and B, using Jefferson's method. Let p_A and p_B denote the populations of A and B, respectively. Show that if the apportionment under Jefferson's method gives all three seats to A and none to B, then more than 75% of the country's population must live in state A. (*Hint*: Show that a Jefferson apportionment of 3 and 0 seats implies that $p_A > 3p_B$.)

69. Consider the problem of apportioning $M = 3$ seats between two states, A and B, using Webster's method. Let p_A and p_B denote the populations of A and B, respectively. Show that if the apportionment under Webster's method gives all three seats to A and none to B, then more than $83\frac{1}{3}\%$ of the country's population must live in state A. (*Hint*: Show that a Webster apportionment of 3 and 0 seats implies that $p_A > 5p_B$.)

70. Consider the problem of apportioning M seats between two states, A and B. Let q_A and q_B denote the standard quotas of A and B, respectively, and assume that these quotas have decimal parts that are not equal to 0.5. Explain why in this case

(a) Hamilton's and Webster's methods must give the same apportionment.

(b) the Alabama or population paradoxes cannot occur under Hamilton's method. [*Hint*: Use the result of (a).]

(c) violations of the quota rule cannot occur under Webster's method. [*Hint*: Use the result of (a).]

71. (a) Explain why, when Jefferson's method is used, any violations of the quota rule must be upper-quota violations.

(b) Explain why, when Adams's method is used, any violations of the quota rule must be lower-quota violations.

(c) Explain why, in the case of an apportionment problem with two states, violations of the quota rule cannot occur under either Jefferson's or Adams's method. [Hint: Use the results of (a) and (b).]

72. Alternate version of Hamilton's method. Consider the following description of an apportionment method:

- **Step 1.** Find each state's standard quota.
- **Step 2.** Give each state (temporarily) its *upper quota* of seats. (You have now given away more seats than the number of seats available.)
- **Step 3.** Let K denote the number of extra seats you have given away in Step 2. Take away the K extra seats from the K states with the smallest fractional parts in their standard quotas.

Explain why this method produces exactly the same apportionment as Hamilton's method.

Lowndes's Method. *Exercises 73 and 74 refer to a variation of Hamilton's method known as Lowndes's method, first proposed in 1822 by South Carolina Representative William Lowndes. The basic difference between Hamilton's and Lowndes's methods is that in Lowndes's method, after each state is assigned the lower quota, the surplus seats are handed out in order of relative fractional parts. (The relative fractional part of a number is the fractional part divided by the integer part. For example, the* ***relative fractional part*** *of 41.82 is* $\frac{0.82}{41} = 0.02$*, and the relative fractional part of 3.08 is* $\frac{0.08}{3} = 0.027$*. Notice that while 41.82 would have priority over 3.08 under Hamilton's method, 3.08 has priority over 41.82 under Lowndes's method because 0.027 is greater than 0.02.)*

73. (a) Find the apportionment of Parador's Congress (Example 4.3) under Lowndes's method.

(b) Verify that the resulting apportionment is different from each of the apportionments found under the other methods discussed in the chapter. In particular, list which states do better under Lowndes's method than under Hamilton's method.

74. Consider an apportionment problem with two states, A and B. Suppose that state A has standard quota q_1 and state B

has standard quota q_2, neither of which is a whole number. (Of course, $q_1 + q_2 = M$ must be a whole number.) Let f_1 represent the fractional part of q_1 and f_2 the fractional part of q_2.

(a) Find values q_1 and q_2 such that Lowndes's method and Hamilton's method result in the same apportionment.

(b) Find values q_1 and q_2 such that Lowndes's method and Hamilton's method result in different apportionments.

(c) Write an inequality involving q_1, q_2, f_1, and f_2 that would guarantee that Lowndes' method and Hamilton's method result in different apportionments.

RUNNING

75. The Hamilton-Jefferson hybrid method. The Hamilton-Jefferson hybrid method starts by giving each state its lower quota (as per Hamilton's method) and then apportioning the surplus seats using Jefferson's method.

(a) Use the Hamilton-Jefferson hybrid method to apportion $M = 22$ seats among four states according to the following populations: A (population 18,000), B (population 18,179), C (population 40,950), and D (population 122,871).

(b) Explain why the Hamilton-Jefferson hybrid method can produce apportionments that are different from both Hamilton and Jefferson apportionments.

(c) Explain why the Hamilton-Jefferson hybrid method can violate the quota rule.

76. Explain why Jefferson's method cannot produce

(a) the Alabama paradox

(b) the new-states paradox

77. Explain why Adams's method cannot produce

(a) the Alabama paradox

(b) the new-states paradox

78. Explain why Webster's method cannot produce

(a) the Alabama paradox

(b) the new-states paradox

PROJECTS AND PAPERS

1 Dean's Method

Dean's method (also known as the *method of harmonic means*) was first proposed in 1832 by James Dean. [This is a different James Dean. Dean (1776–1849) was a Professor of Astronomy and Mathematics at Dartmouth College and the University of Vermont.] Dean's method is a divisor method similar to Webster's method but with a slightly different set of rules for rounding the quotas. In this project you are to prepare a report on Dean's method. Your report should include (1) a discussion of the *harmonic mean*, including some of its properties and applications; (2) a description of Dean's method; (3) a discussion of how Dean's method can be implemented using trial and error; and (4) a comparison of Dean's method with Webster's method with an example showing that the two methods can produce different apportionments.

2 Apportionment Methods and the 2000 Presidential Election

After tremendous controversy over hanging chads and missed votes, the 2000 presidential election was decided in the United States Supreme Court, with George W. Bush getting the disputed electoral votes from Florida and thus beating Al Gore by a margin of four electoral votes. Ignored in all the controversy was the significant role that the choice of apportionment method plays in a close presidential election. Would things have turned out differently if the House of Representatives had been apportioned under a different method?

In this project, you are asked to analyze and speculate how the election would have turned out had the House of Representatives (and thus the Electoral College) been apportioned under (1) Hamilton's method, (2) Jefferson's method, and (3) Webster's method.

Notes: (1) The 2000 Electoral College was based on the 1990 U.S. Census. (Remember to add 2 to the seats in the House to get each state's Electoral College votes!) You can go to *www.census.gov/main/www//cen1990.html* for the 1990 state population figures. (2) You can use the Apportionment applets in MyMathLab* to do the calculations.

3 Rank Index Implementations of Divisor Methods

From a computational point of view, the divisor methods of Jefferson, Adams, and Webster can be implemented using a priority system (called a **rank index**) that gives away seats one by one according to a table of values specifically constructed for this purpose. Each of the methods uses a slightly different set of rules to construct the rank index for the seats. In this project you are to describe how to construct the rank index for each of the three divisor methods studied in this chapter (Jefferson, Adams, and Webster). You should use a simple example to illustrate the procedure for each method.

*MyMathLab code required.