

DIMENSION OF ESCAPING GEODESICS

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ABSTRACT. Suppose $M = \mathbb{B}/G$ is a hyperbolic manifold. Consider the set of escaping geodesic rays $\gamma(t)$ originating at a fixed point p of the manifold M , i.e. $\text{dist}(\gamma(t), p) \rightarrow \infty$. We investigate those escaping geodesics which escape at the fastest possible rate, and find the Hausdorff dimension of the corresponding terminal points on the boundary of \mathbb{B} .

In dimension 2, for a geometrically infinite Fuchsian group, if the injectivity radius of $M = \mathbb{B}/G$ is bounded above and away from zero, then these points have full dimension. In dimension 3, when G is a geometrically infinite and topologically tame Kleinian group, if the injectivity radius of $M = \mathbb{B}/G$ is bounded away from zero, the dimension of these points is 2, which is again maximal.

1. INTRODUCTION

Consider G , a discrete, torsion free group of isometries of the hyperbolic metric on the hyperbolic three ball \mathbb{B} ; i.e. a Kleinian group. Passing to the quotient \mathbb{B}/G by identification of the G -equivalent points we obtain the quotient space M , which is a manifold. Suppose this group G is non-elementary, and denote its limit set by Λ .

A point x on the boundary of the ball is a non-conical point if there is a geodesic ray ending at x so that the projection of this geodesic will eventually leave any compact set and tend to the ideal boundary. Among these points there is a subset that escapes to the ideal boundary at the fastest possible rate; these are called deep points. The original definition of deep points is due to McMullen ([14]). A point is a *deep point* if there is a geodesic ray $\gamma : [0, \infty) \rightarrow C(\Lambda)$ in the convex hull parameterized by arclength and terminating at x , so that for some $\delta > 0$

$$\frac{\text{dist}(\gamma(t), \partial C(\Lambda))}{t} \geq \delta$$

for all t , i.e. the depth of γ inside the convex hull of the limit set Λ increases linearly with the hyperbolic length. (All the necessary definitions will be given later in the text.)

We can generalize this notation by taking any Lipschitz function $\phi(t) : [1, \infty) \rightarrow [1, \infty)$ with the property of $\lim_{t \rightarrow \infty} \phi(t) = \infty$. We fix a point $z_0 \in M = \mathbb{B}/G$, and

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consider the set of geodesic rays starting at z_0 parameterized by the hyperbolic arclength. Define the set of geodesics in the convex core which escape at a rate ϕ as

$$\Gamma_\phi^C = \left\{ \gamma : \frac{1}{C} \leq \frac{\text{dist}(\gamma(t), z_0)}{\phi(t)} \leq C \right\}.$$

Let Λ_ϕ^C denote the terminating points of the geodesics in Γ_ϕ^C , and let $\Lambda_\phi = \bigcup_C \Lambda_\phi^C$.

The main theorem of this paper is the following:

Theorem 1.1. *Suppose G is a geometrically infinite, topologically tame Kleinian group, $M = \mathbb{B}/G$ has injectivity radius bounded away from zero and there is a Green's function on M . Let $\phi(t) : [1, \infty) \rightarrow [1, \infty)$ be a Lipschitz function satisfying $\lim_{t \rightarrow \infty} \phi(t) = \infty$; then the Hausdorff dimension of Λ_ϕ is 2.*

The relevant definitions will be given later in Sections 2, 3 and 4. The idea of the proof is as follows: we can find a positive harmonic function u on the manifold M which tends to 0 in the geometrically finite ends of M (Lemma 4.1). Lifting u up to the covering space \mathbb{B} , we get a hyperbolic harmonic function U on the ball \mathbb{B} . This hyperbolic harmonic function is a Poisson integral of some positive measure μ , which is supported on the limit set. Using this measure μ , we construct a Bloch martingale $\{f_n\}$ on the dyadic squares Q of length 2^{-n} by defining f_n as

$$f_n(x) = \frac{\mu(Q(x))}{m(Q(x))}.$$

With the help of a technical lemma (Lemma 5.2) we can find a Cantor set, which has Hausdorff dimension two (Lemma 3.3), on which the martingale grows approximately at the same rate as the given Lipschitz function ϕ , i.e.

$$\frac{1}{C} \leq \left| \frac{f_n(Q)}{\phi(n)} \right| \leq C.$$

The martingale $f_n(x)$ for $x \in Q$ has bounded distance from the harmonic function U on the top of the Carleson square drawn over Q (Lemma 4.2); therefore

$$\frac{1}{C} \leq \left| \frac{U(z_Q)}{\phi(n)} \right| \leq C.$$

Finally, $U(z)$ gives the distance approximately from $\gamma(t)$ to the base point, which gives an estimate for $\text{dist}(\gamma(t), \partial C(\Lambda))$ on manifolds specified in the main theorem.

An analogous theorem can also be given for Fuchsian groups:

Theorem 1.2. *Suppose G is a geometrically infinite Fuchsian group, $M = \mathbb{B}/G$ has injectivity radius bounded and bounded away from zero, and there is a Green's function on M . Let $\phi(t) : [1, \infty) \rightarrow [1, \infty)$ be a Lipschitz function satisfying $\lim_{t \rightarrow \infty} \phi(t) = \infty$; then the Hausdorff dimension of Λ_ϕ is 1.*

Fernández and Melián in [9] extensively studied the size of the set of escaping geodesics starting at a point of the hyperbolic surface. A geodesic ray $\gamma(t)$ originating at p is called *escaping* if $\lim_{t \rightarrow \infty} \text{dist}(\gamma(t), p) = +\infty$. These rays may leave the convex core. It is known that if there is a Green's function on the surface, then the set of escaping geodesics has full measure. If there is no Green's function, then the Hausdorff dimension of the terminal points is still 1.

Taking $\phi(t) = t$ in Theorems 1.1 and 1.2 we get the dimension of deep points in sets described in the theorem above.

Corollary 1.3. *If G is a geometrically infinite, topologically tame Kleinian group and $M = \mathbb{B}/G$ has injectivity radius bounded away from zero and has a Green's function, then the deep points have dimension 2.*

2. DEFINITIONS AND NOTATION

Let $M(\bar{\mathbb{R}}^d)$ denote the orientation preserving Möbius transformations in the d -dimensional extended space $\bar{\mathbb{R}}^d = \mathbb{R}^d \cup \{\infty\}$. The subgroup of $M(\bar{\mathbb{R}}^d)$ which preserves the upper half-plane $\mathbb{H} = \{x \in \bar{\mathbb{R}}^d : x_d > 0\}$ or the unit ball $\mathbb{B} = \{x \in \bar{\mathbb{R}}^d : |x| < 1\}$ will be denoted by $M(\mathbb{H})$ and $M(\mathbb{B})$, respectively. A discrete group G of $M(\mathbb{B})$ in dimension 3 is called a *Kleinian group*. A *Fuchsian group* is a Kleinian group that stabilizes a round disc on $\partial\mathbb{B}$, the sphere at infinity. In this paper we consider only *non-elementary* groups, that is, G has no finite orbit in $\mathbb{H}^3 = \{x \in \bar{\mathbb{R}}^3 : x_3 > 0\}$.

If G is a discrete subgroup of $M(\mathbb{B})$, the orbit $G(a)$ of any point $a \in \mathbb{B}$ can accumulate only on the boundary of \mathbb{B} . So we call a point $x \in S = \partial\mathbb{B}$ a *limit point*, if there is an orbit $G(a)$ accumulating at x . The *limit set* is the set of limit points and is denoted by $\Lambda(G)$ or simply by Λ . The complementary set $S \setminus \Lambda$ of Λ is called the *ordinary set*, and is denoted by Ω ([3]).

Let G be a Kleinian group. Then the quotient space Ω/G , which is obtained from the ordinary set of G by identifying equivalent points under the mappings of G , is a marked (possibly disconnected) Riemann surface ([13]). If Ω/G is a finite marked Riemann surface (i.e. a finite union of compact surfaces, each with at most a finite number of punctures), then G is called *analytically finite*. The Ahlfors finiteness theorem shows that G is analytically finite if it is finitely generated.

A Möbius group G is called *geometrically finite* if some convex fundamental polyhedron has finitely many faces. In dimensions 2 and 3 the standard definition of geometric finiteness is that the Dirichlet region must have finitely many faces. It is known that this criterion implies that every Dirichlet region and every convex fundamental polyhedron has finitely many faces ([3], [15]). Moreover, geometric finiteness implies that the group is finitely generated, and therefore analytically finite.

The *convex hull* of $\Lambda \subset S = \partial\mathbb{B}$, denoted by $C(\Lambda)$, is the smallest convex subset of \mathbb{B} containing all geodesics with both endpoints in Λ . The *convex core* of a hyperbolic manifold $M = \mathbb{B}/G$ is given by the quotient $C(\Lambda)/G$ and denoted by $C(M)$. For $x \in M$ the *injectivity radius*, $\text{inj}(x)$, is half the distance between the two closest distinct lifts of x to \mathbb{B} . In the theorem we assume that the injectivity radius is bounded away from zero uniformly on M , which in dimensions 2 and 3 implies that G has no parabolic elements.

A Kleinian group is called *topologically tame*, if the corresponding quotient manifold $M = \mathbb{B}/G$ is homeomorphic to the interior of a compact 3-manifold with boundary. This implies that the convex core $C(M)$ consists of a compact piece and a finite number of ends E_j , which are topologically equivalent to $S \times \mathbb{R}_+$ for some compact surface S . D. Calegari and D. Gabai in [6] and I. Agol in [1] proved the Marden conjecture, where all complete hyperbolic 3-manifolds with finitely generated fundamental group are topologically tame. We note here that in [7] Canary

showed that topological tameness is equivalent to analytical tameness in dimension 3. Moreover, if G is topologically tame, then there is an upper bound for the injectivity radius inside the convex core.

In the Introduction we gave the definition of a deep point defined by a geodesic ray in the convex hull of Λ . An equivalent definition can also be given on the quotient manifold, as in [4]. A point $x \in \Lambda$ is deep if the geodesic ray γ ending at x satisfies

$$\frac{\text{dist}(\tilde{\gamma}(t), M \setminus C(M))}{t} \geq \delta > 0$$

for all $t \geq t_0$, where $\tilde{\gamma}$ denotes the curve on the quotient space which corresponds to γ .

3. DYADIC MARTINGALE AND HAUSDORFF DIMENSION

An n th generation dyadic cube in \mathbb{R}^d is

$$Q_n = \{x = (x_1, x_2, \dots, x_d) : a_i \leq x_i < a_i + 2^{-n}, 1 \leq i \leq d\}$$

where $a = (a_1, a_2, \dots, a_d)$, the corner of the cube, has coordinates in the form $a_i = \frac{m_i}{2^n}$ with an integer m_i . The collection of these dyadic cubes is denoted by \mathcal{D}_n . For any given point $x \in \mathbb{R}^d$ let $Q_n(x)$ denote that unique n th generation dyadic cube from \mathcal{D}_n which contains the point x , and let $|Q_n|$ denote the side-length of Q_n . The m th generation descendants of Q_n are the dyadic sub-cubes of Q_n with side-length of $2^{-m}|Q_n|$. There are 2^{md} of them.

Suppose Q_0 is a unit cube in \mathbb{R}^d . Then a sequence of functions $\{f_n\}_{n=0}^\infty$ is said to be a *dyadic martingale* on Q_0 if

- (1) f_n is measurable on each $Q_n \in \mathcal{D}$,
- (2) $\frac{1}{|Q_n|} \int_{Q_n} f_n < \infty$,
- (3) $\frac{1}{|Q_m|} \int_{Q_m} f_n = f_m$ for all $m < n$.

In addition to this usual definition, we also require that f_n must be constant on the n th generation dyadic cubes. Since in this paper we use only dyadic martingales, so we often omit the “dyadic” attribute.

There is a standard way to construct a martingale from a finite measure. If a finite measure μ is given on Q_0 , then the functions

$$(3.1) \quad f_n(x) = \frac{\mu(Q_n(x))}{|Q_n(x)|_d}$$

define a dyadic martingale, where $|Q_n(x)|_d$ (or just $|Q_n(x)|$ if the notation is clear from the text) denotes the d -dimensional Lebesgue measure of $Q_n(x)$. We define the *martingale differences* as $\Delta f_n(x) = f_{n+1}(x) - f_n(x)$ and the *martingale square function* as

$$S_f(x) = \left(\sum_{n=1}^{\infty} \|\chi_{Q_n(x)} \Delta f_n\|_{\infty}^2 \right)^{1/2}.$$

A martingale is called *Bloch* if $\sup_n \|\Delta f_n\|_{\infty} < \infty$. If $\{f_n\}$ is an L^1 -bounded martingale, then f_n converges a.e. to a function f with $\|f\|_1 < \infty$. For more results on the convergence of martingales, you may see [11]. We will need the following two estimates for dyadic martingales.

Lemma 3.1. *Let f_n be a dyadic martingale on $Q_0 \subset \mathbb{R}^d$ with limit function f . Suppose $\|S_f\|_\infty < \infty$. Then for $\lambda \geq 0$,*

$$|x \in Q_0 : f(x) - f_0(x) \geq \lambda| \leq \exp\left(-\frac{\lambda^2}{2\|S_f\|_\infty^2}\right).$$

The proof of this lemma is due to Herman Rubin and can be found in the paper of Chang-Wilson-Wolff [8, Theorem 3.1].

Lemma 3.2. *Suppose μ is a probability measure on X . Suppose F is a measurable, real valued function on X so that $\int_X F d\mu = 0$ and $\|F\|_4 \leq B\|F\|_2$. Then*

$$\mu\left(\left\{x : F(x) \leq -\frac{1}{\sqrt{8}B^2}\|F\|_2\right\}\right) \geq \frac{1}{64B^{12}}.$$

The proof is given in [4].

Suppose ϕ is an increasing, continuous function from $[0, \infty)$ to itself such that $\phi(0) = 0$. For a given set E we define the *Hausdorff content* as

$$\mathcal{H}_\infty^\phi(E) = \inf\left\{\sum \phi(r_j) : E \subset \cup_j D(x_j, r_j)\right\},$$

where $D(x_j, r_j)$ denotes a ball of radius r_j centered at x_j . Especially, if $\phi(t) = t^\alpha$ we denote \mathcal{H}_∞^ϕ by $\mathcal{H}_\infty^\alpha$. The *Hausdorff dimension* of a set E is

$$\dim_H(E) = \inf\{\alpha : \mathcal{H}_\infty^\alpha(E) = 0\}.$$

For more details and examples on Hausdorff dimension you may see [5].

In the proof of our theorem we will also need the following lemma.

Lemma 3.3. *Suppose E_n is a union of closed dyadic cubes of generation k_n so that $E_0 \supset E_1 \supset E_2 \supset \dots$ and there are constants N and ϵ with*

- (1) $|k_n - k_{n+1}| = N$ for all n .
- (2) If $Q \in E_n$ is generation k_n , then $|E_{n+1} \cap Q|_d \geq \epsilon|Q|_d$.

If $E = \bigcap_n E_n$, then $\dim_H(E) \geq d - C(N, \epsilon)$, where $C(N, \epsilon) \rightarrow 0$ whenever $\epsilon > 0$ is fixed and $N \rightarrow \infty$.

The proof of this lemma can be found in [10], [12] and [16].

4. THE HYPERBOLIC SPACE AND HARMONIC FUNCTIONS

The unit ball \mathbb{B} in \mathbb{R}^n is the disc model for the n -dimensional hyperbolic space equipped with the hyperbolic metric

$$(4.1) \quad d\rho = \frac{2|dx|}{1 - |x|^2}.$$

An alternative model of the hyperbolic n -space is the upper half-plane $\mathbb{H} = \{x = (x_1, x_2, \dots, x_n) : x_n > 0\} \subset \mathbb{R}^n$ equipped with the metric

$$(4.2) \quad d\rho = \frac{|dx|}{x_n}.$$

Using the hyperbolic metric defined in $\mathbb{B} = \{x \in \mathbb{R}^n : |x| < 1\}$ by (4.1) we may construct the hyperbolic volume and area element, the normal derivative and

gradient in the hyperbolic ball as

$$\begin{aligned} dV_H &= \frac{2^n dx_1 dx_2 \cdots dx_n}{(1 - |x|^2)^n}, \\ d\sigma_H &= \frac{2^{n-1} d\sigma}{(1 - |x|^2)^{n-1}}, \\ \frac{\partial v}{\partial n_H} &= \frac{1 - |x|^2}{2} \frac{\partial v}{\partial n}, \\ \nabla_H u &= \frac{1 - |x|^2}{2} \nabla u. \end{aligned}$$

On the upper half-plane $\mathbb{H} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n, x_n > 0\}$ these are

$$\begin{aligned} dV_H &= \frac{dx_1 \cdots dx_n}{x_n^n}, \\ d\sigma_H &= \frac{d\sigma}{x_n^{n-1}}, \\ \frac{\partial v}{\partial n_H} &= x_n \frac{\partial v}{\partial n}, \\ \nabla_H u &= x_n \nabla u, \end{aligned}$$

respectively. A more detailed description can be found in [2] and [15].

The hyperbolic Laplace-Beltrami operator for the unit ball $\mathbb{B} \subset \mathbb{R}^n$ is given by

$$\Delta_H = \frac{(1 - r^2)^2}{4} \left[\Delta + \frac{2(n - 2)r}{1 - r^2} \frac{\partial}{\partial r} \right],$$

where $r = |x|$. On the upper half-plane this is

$$\Delta_H = x_n^2 \left[\Delta - \frac{n - 2}{x_n} \frac{\partial}{\partial x_n} \right].$$

A function f is called *hyperbolically harmonic* if it satisfies the hyperbolic Laplace equation, $\Delta_H f = 0$.

We define the Green's function on a quotient manifold M as follows. F is a *Green's function* on M with a pole at the projection of a point a , if there exists a function $f : \mathbb{B} \setminus \{G(a)\} \rightarrow \mathbb{R}$ such that the projection of f is F and the following are true for f :

- f is a hyperbolic harmonic function on $\mathbb{B} \setminus \{G(a)\}$,
- $f \circ g = f$ for all $g \in G$,
- $\lim_{z \rightarrow a} (f(z) - \frac{1}{z-a})$ exists, i.e. f has singularity $\frac{1}{z-a}$ at the point a ,
- f is the smallest positive function with these properties.

The hyperbolic version of Green's formula is

$$\int_D (u \Delta_H v - v \Delta_H u) dV_H = \int_{\partial D} \left(u \frac{\partial v}{\partial n_H} - v \frac{\partial u}{\partial n_H} \right) d\sigma_H.$$

We will need the following existence theorem, which was proven in [4].

Lemma 4.1. *Suppose G is topologically tame, geometrically infinite, $M = \mathbb{B}/G$ has an injectivity radius bounded below by $\epsilon > 0$ and Green's function $G(w, z)$ exists on M . Then there exists a positive harmonic function U on M such that*

$$\sup_{z \in M} |\nabla U(z)| \leq 1$$

and U tends to zero in the geometrically finite ends of M . If, in addition, G is topologically tame, then for any $a_0 > 0$ there are constants a_1 and a_2 so that

$$\int_{B(z, a_1)} |\nabla U|^2 dV \geq a_2,$$

for every z such that $\text{dist}(z, C(M)) \leq a_0$. Moreover, $U(z)$ tends to $+\infty$ in the geometrically infinite ends as $\text{dist}(z, \partial C(M)) \rightarrow \infty$.

If Q is a cube in \mathbb{R}^n , then $\hat{Q} = Q \times [0, \ell(Q)]$ is called the Carleson cube in \mathbb{R}_+^{n+1} with base Q , and let z_Q denote the center of \hat{Q} .

Lemma 4.2. *Suppose U on \mathbb{R}_+^{n+1} is the hyperbolic Poisson integral of the positive measure μ and satisfies $|\nabla_H U(z)| \leq 1$. For a square $Q \in \mathbb{R}^n$, let $Q_t = \{\underline{x}, t\} : \underline{x} \in Q\}$. Then there is an $A < \infty$ so that*

$$\left| U(z_Q) - \frac{1}{|Q|} \int_Q U(\underline{x}, t) d\underline{x} \right| \leq A,$$

for any $0 < t \leq \ell(Q)$, where $\ell(Q)$ denotes the side-length of Q , and

$$\left| U(z_Q) - \frac{1}{|Q|} \int_Q d\mu \right| \leq A.$$

The proof of this lemma in dimension $n = 2$ was given in [4], and it can be proven in higher dimension on a similar way as in [10].

5. TWO LEMMAS ON MARTINGALES

Define a martingale on the dyadic cubes using the positive measure μ described in Lemma 4.1 defined by equation (3.1). According to Lemma 4.2, there exists a constant A so that $|U(z_{Q_n(x)}) - f_n(x)| \leq A$. To prove our main theorem we will need the following two lemmas for dyadic martingales.

Lemma 5.1. *Suppose μ is a positive measure on the cube $[0, 1]^d$, $d \geq 1$, so that the corresponding dyadic martingale defined by $f_n(x) = \frac{\mu(Q_n(x))}{|Q_n(x)|}$ is Bloch and $\frac{1}{|Q_n|} \|\Delta f_n\|_2^2 \geq \delta > 0$ whenever $f_n(x) \geq 1$ on $Q_n(x)$. We claim that there is an $\epsilon > 0$ and $M < \infty$ so that for any sufficiently large n , there is a constant C for which the following holds. Let Q be any dyadic cube, and let f_Q denote the function in the martingale defined by Q , i.e. $f_Q = \frac{\mu(Q)}{|Q|}$ on Q . Suppose $f_Q \geq C$. Then among the 2^{dn} n th generation descendants of Q , at least $\epsilon 2^{dn}$ satisfy $Mn \geq f_{Q'} - f_Q \geq 1$, and at least $\epsilon 2^{dn}$ satisfy $-Mn \leq f_{Q'} - f_Q \leq -1$.*

Proof. Suppose $\sup_n \|\Delta f_n\|_\infty = L < \infty$ and $\frac{1}{|Q_n|} \|\Delta f_n\|_2^2 \geq \delta > 0$ whenever $f_n(x) \geq 1$, and fix an ϵ with $0 < \epsilon \leq \min\{\frac{\delta^6}{2^{16}L^{12}}, 1\}$. By an appropriate scaling we may assume that $|Q| = 1$. Then the martingale square function for the sequence $\{f_0, f_1, \dots, f_n\}$ is

$$(5.1) \quad S_f(x) = \left(\sum_{j=0}^{n-1} \|\chi_{Q_j(x)} \Delta f_j\|_\infty^2 \right)^{1/2} \leq \sqrt{n}L.$$

Let $F = \Delta f_0 + \dots + \Delta f_{n-1}$, and suppose that $n > \frac{64L^4}{\delta^3}$ and that $f_Q \geq 1 + nL = C$. Then $f_j \geq 1$ for all $0 \leq j \leq n-1$, so $\|\Delta f_j\|_2^2 \geq \delta$ for all $0 \leq j \leq n-1$. The system $\{\Delta f_j\}_{j=0}^{n-1}$ is orthogonal, therefore

$$(5.2) \quad nL^2 \geq \|F\|_2^2 = \sum_{j=0}^{n-1} \|\Delta f_j\|_2^2 \geq n\delta.$$

Let $\lambda(t) = |\{x : |F(x)| > t\}|$ define the distribution function of $|F|$. Then

$$(5.3) \quad \int |F|^p = p \int_0^\infty t^{p-1} \lambda(t) dt,$$

and by Lemma 3.1, $\lambda(t) \leq e^{-\frac{t^2}{2\|S_F\|_\infty^2}} \leq e^{-\frac{t^2}{2L^2n}}$. Therefore

$$(5.4) \quad \begin{aligned} \|F\|_4^4 &= \int |F|^4 = 4 \int_0^\infty t^3 \lambda(t) dt \\ &\leq 4 \int_0^\infty t^3 e^{-\frac{t^2}{2L^2n}} dt \\ &= 8L^4 n^2 \int_0^\infty y e^{-y} dy \\ &= 8L^4 n^2. \end{aligned}$$

Hence $\|F\|_4 \leq \sqrt[4]{8L^4 n^2} = B\delta\sqrt{n} \leq B\|F\|_2$ with the constant $B = \frac{\sqrt[4]{8}L}{\sqrt{\delta}}$. Now we can apply Lemma 3.2 so

$$(5.5) \quad \mu\left(\left\{x \in Q : F(x) \leq -\frac{1}{\sqrt{8}B^2}\|F\|_2\right\}\right) \geq \frac{1}{64B^{12}}.$$

Using the fact that $\|F\|_2 \geq \sqrt{n\delta}$ and the assumptions that $\sqrt{n} > \frac{8L^2}{\sqrt{\delta^3}} = \frac{\sqrt{8}}{\sqrt{\delta}}B^2$ we get that

$$(5.6) \quad \begin{aligned} \mu(\{x \in Q : F(x) \leq -1\}) &\geq \mu\left(\left\{x \in Q : F(x) \leq -\frac{1}{\sqrt{8}B^2}\sqrt{n\delta}\right\}\right) \\ &\geq \mu\left(\left\{x \in Q : F(x) \leq -\frac{1}{\sqrt{8}B^2}\|F\|_2\right\}\right) \\ &\geq \frac{1}{64B^{12}} \\ &= \frac{\delta^6}{2^{15}L^{12}} \\ &\geq 2\epsilon. \end{aligned}$$

Switching F with $-F$, with the same assumptions, we get

$$(5.7) \quad \mu(\{x \in Q : F(x) \geq 1\}) \geq 2\epsilon.$$

Next, consider the following subsequence $\{f_0, \dots, f_n\}$. By Lemma 3.1, for a positive constant $M \geq L\sqrt{-2 \ln \epsilon}$,

$$\begin{aligned}
 (5.8) \quad |\{x \in Q : f_n(x) - f_0(x) \geq nM\}| &\leq \exp\left(\frac{-n^2 M^2}{2\|S_f\|_\infty^2}\right) \\
 &\leq \exp\left(\frac{-nM^2}{2L^2}\right) \\
 &\leq \left(\exp\left(-\frac{M^2}{2L^2}\right)\right)^n \\
 &\leq \epsilon
 \end{aligned}$$

for every $n \geq 1$ and $\epsilon < 1$. Repeating this argument for $\{-f_0, \dots, -f_n\}$ we get that

$$(5.9) \quad |\{x \in Q : f_n(x) - f_0(x) \leq -nM\}| \leq \epsilon.$$

Therefore, for every sufficiently large n there is a constant $C = 1 + nL$ so that if $f_Q \geq C$, then

$$\mu(\{x \in Q : 1 \leq F(x) = f_n(x) - f_0(x) \leq nM\}) \geq \epsilon$$

and

$$\mu(\{x \in Q : -1 \geq F(x) = f_n(x) - f_0(x) \geq -nM\}) \geq \epsilon.$$

In other words, among all of the 2^{dn} n th generation descendants of Q , at least $\epsilon 2^{dn}$ satisfy the inequality $Mn \geq f_{Q'} - f_Q \geq 1$, and at least $\epsilon 2^{dn}$ of them satisfy $-Mn \leq f_{Q'} - f_Q \leq -1$. \square

Lemma 5.2. *Suppose G is a topologically tame and geometrically infinite Kleinian group, so that $M = \mathbb{B}/G$ has injectivity radius bounded below by some positive epsilon and there exists a Green's function on M . Let U be a positive harmonic function on M for which $\sup_{z \in M} |\nabla_H U(z)| \leq 1$, and let $\{f_m\}$ denote the corresponding martingale as defined by (3.1). Then this martingale $\{f_m\}$ has bounded differences away from zero in the L_2 norm, whenever its value is larger than some fixed constant C .*

Proof. Suppose $f_m > C$ on the dyadic cube Q_m . From Lemma 4.2 we know that $|U(z_Q) - f_m| \leq A$, where z_Q denotes the center of the Carleson square in \mathbb{R}_+^{n+1} with base Q_m . Since G is topologically tame and $\text{inj}(z) \geq \epsilon > 0$, the convex core $C(M)$ can be written as a compact part and a finite number of ends E_j , each of which is topologically equivalent to $S \times \mathbb{R}^+$ for some compact surface S ([7]). We may suppose that we are already in such an end.

First, we will show that for the given constant A , there exists a constant L , so that for all $v \in C(M)$ with $U(v) \geq C$ we can find another point w with $\rho(v, w) \leq L$ and $|U(v) - U(w)| \geq 6A$. Lemma 4.1 says that there exist r and a so that

$$(5.10) \quad \int_{B(z,r)} |\nabla U|^2 dV \geq a$$

for every $z \in C(M)$. Consider a geodesic ray on M originating at the point v and going to infinity in the convex core. We may put disjoint balls of radius r along this geodesic, say N balls, and denote w the endpoint, so $\rho(v, w) = 2rN$. Cut E_j at v and at w , and call these surfaces Σ_1 and Σ_2 , respectively, and let T denote

the part of E_j between these cuts. We may also assume that $U(v) = 0$. Green's Theorem says that

$$(5.11) \quad \int_T f \Delta g - g \Delta f dV_H = \int_{\Sigma_1 \cup \Sigma_2} f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} d\sigma.$$

Let $f = 1$ and $g = U^2$; then

$$(5.12) \quad \int_T \Delta(U^2) dV = \int_{\Sigma_1 \cup \Sigma_2} \frac{\partial(U^2)}{\partial n} d\sigma.$$

By elementary calculations we get that

$$(5.13) \quad \int_T |\nabla U|^2 dV = \int_{\Sigma_1 \cup \Sigma_2} U \frac{\partial U}{\partial n} d\sigma.$$

Using Lemma 4.1, we can estimate the left-hand side of (5.13) by

$$(5.14) \quad \int_T |\nabla U|^2 \geq Na.$$

For the estimation of the right-hand side of (5.13) we can use that $|\nabla U| \leq 1$, so

$$(5.15a) \quad \int_{\Sigma_1} U \frac{\partial U}{\partial n} d\sigma \leq \text{diam}(\Sigma_1) \text{area}(\Sigma_1),$$

$$(5.15b) \quad \int_{\Sigma_2} U \frac{\partial U}{\partial n} d\sigma \leq (U(w) + \text{diam}(\Sigma_2)) \text{area}(\Sigma_2).$$

Since $E = S \times \mathbb{R}^+$, $\text{diam}(\Sigma_i) \leq D$ and $\text{area}(\Sigma_i) \leq S$ along the entire end E_j . Using the estimations (5.14) and (5.15) in (5.13), we get that

$$(5.16) \quad Na \leq DS + (U(w) + D)S,$$

and so

$$(5.17) \quad \frac{Na}{S} - 2D \leq U(w).$$

Therefore, we can choose a uniform N large enough so that $|U(v) - U(w)| \geq 6A$ and let $L = 2rN$.

Next, we show that there is a point w such that $|U(z_Q) - U(w)| \geq 3A$ while $\rho(z_Q, w) \leq 3L$. We start at the point z_Q on the Carleson square and go straight down toward the boundary by hyperbolic distance $2L$; we call this point v . As we have just shown above, there exists a w such that $\rho(v, w) \leq L$ while $|U(v) - U(w)| \geq 6A$, which means that either $|U(z_Q) - U(v)| \geq 3A$ or $|U(z_Q) - U(w)| \geq 3A$. Assume the latter is true.

Finally, we show that there is a subfamily $\{f_{m_i}\}$ in the original martingale sequence with bounded differences away from zero in L_2 -norm while $m_{i+1} - m_i \leq 3L$ for all i . From Lemma 4.2, $|U(z_Q) - f_m| \leq A$ and we can find a w such that $|U(z_Q) - U(w)| \geq 3A$, but $\rho(z_Q, w) \leq 3L$. We may also assume that w is in the middle of a Carleson square, since $|\nabla U| \leq 1$. This Carleson square is different from the original Q_m , call it $Q_{m'}$, and let $f_{m'}$ be the martingale function determined by the size of this square. Then from Lemma 4.2 we have that $|U(w) - f_{m'}| \leq A$ and $|U(z_Q) - f_m| \leq A$, while $|U(z_Q) - U(w)| \geq 3A$. So $|f_m - f_{m'}| \geq A$ on $Q_{m'}$, while $|m - m'| \leq 3L$. Therefore

$$(5.18) \quad \frac{1}{|Q_m|} \int_{Q_m} |f_m - f_{m'}|^2 dx \geq \frac{1}{|Q_m|} A^2 \frac{|Q_m|}{2^{3L}} = \delta > 0. \quad \square$$

6. THE PROOF OF THE THEOREM

Suppose G is a topologically tame, geometrically infinite Kleinian group and the quotient manifold $M = \mathbb{B}/G$ has injectivity radius bounded away from zero. This implies that G has no parabolic elements. Suppose $\phi(t) : [1, \infty) \rightarrow [1, \infty)$ is a Lipschitz function, i.e.

$$|\phi(s) - \phi(t)| \leq B|s - t|$$

for some $B < \infty$, and satisfies $\lim_{t \rightarrow \infty} \phi(t) = \infty$. Fix a point $z_0 \in M$ and consider the set of geodesic rays starting at z_0 parameterized by hyperbolic arclength. Define the set of geodesics in the convex core which escape at rate ϕ as

$$\Gamma_\phi^C = \left\{ \gamma : C^{-1} \leq \frac{\text{dist}(\gamma(t), z_0)}{\phi(t)} \leq C \right\}.$$

Let Λ_ϕ^C denote the terminal points of these geodesics, and let $\Lambda_\phi = \bigcup_C \Lambda_\phi^C$.

Theorem 6.1. *Suppose G is a geometrically infinite, topologically tame Kleinian group and $M = \mathbb{B}/G$ has injectivity radius bounded away from zero and there is a Green's function on M . Let $\phi(t) : [1, \infty) \rightarrow [1, \infty)$ be a Lipschitz function satisfying $\lim_{t \rightarrow \infty} \phi(t) = \infty$; then $\dim_{\mathbb{H}}(\Lambda_\phi) = 2$.*

The analogous theorem for Fuchsian groups:

Theorem 6.2. *Suppose G is a geometrically infinite Fuchsian group, $M = \mathbb{B}/G$ has bounded injectivity radius which is also bounded away from zero and there is a Green's function on M . Let $\phi(t) : [1, \infty) \rightarrow [1, \infty)$ be a Lipschitz function satisfying $\lim_{t \rightarrow \infty} \phi(t) = \infty$; then $\dim_{\mathbb{H}}(\Lambda_\phi) = 1$.*

Proof of Theorem 6.1. By Lemma 4.1 there exists a positive harmonic function U on M with $\sup_{z \in M} |\nabla U(z)| \leq 1$ and U tends to zero in the geometrically finite ends of M . This U lifts to a hyperbolic harmonic function (which we will also call U) on \mathbb{B} , and this function is a Poisson integral of some positive measure μ supported on the limit set. Consider the corresponding dyadic Bloch martingale $\left\{ f_Q(x) = \frac{\mu(Q(x))}{|Q(x)|} \right\}_{Q \in \mathcal{D}}$.

Using Lemma 5.2 we may pass to a subsequence of $\{f_Q\}$ for which the martingale differences are bounded away from zero whenever the value of the martingale is not less than a constant C . Notice that even if we work with this subsequence we can still use Lemma 5.1 because there are upper and lower bounds for the number of generations we skip. To simplify our indexes we will suppose that $\{f_Q\}$ itself is a Bloch martingale with differences bounded away from zero whenever the value is not less than C .

Using Lemma 5.1 we can create a Cantor set $\{E_l\}$ of nested dyadic cubes, where the dyadic martingale is comparable to the function ϕ . As in Lemma 5.1 find appropriate $\epsilon > 0$, $M < \infty$, and fix a sufficiently large N and the corresponding constant C . Since U tends to infinity on the geometrically infinite ends we may suppose that $f_Q \geq C$, except maybe for finitely many generations of cubes.

First, notice that we may suppose that the function ϕ is Lipschitz with a Lipschitz constant $\frac{1}{N}$, i.e. $|\phi(x) - \phi(y)| \leq \frac{1}{N}|x - y|$. In case $|\phi(x) - \phi(y)| \leq B|x - y|$ with a larger constant B than $\frac{1}{N}$, we can rescale our function by choosing $\Phi(x) = \frac{1}{BN}\phi(x)$.

Define E_0 as the collection of those largest cubes Q where $f_Q \geq C$ and let k_0 be the number which denotes the generation of these cubes. Then there is a positive constant D so that $|f_{k_0}(Q) - \phi(k_0)| \leq D$ on all $Q \in E_0$. We may also assume that $D \geq MN$. We define the sets $\{E_n\}$ inductively as follows. Suppose we already have the set E_n defined, and the quotient (or the difference) of f_{k_l} and $\phi(k_l)$ is bounded on all the previous sets, say $\frac{1}{D} \leq |f_{k_l} - \phi(k_l)| \leq D$ for all $l \leq n$. Let $k_{n+1} = k_n + N$. Since ϕ is a Lipschitz function with constant $\frac{1}{N}$, so $\phi(k_n) - 1 \leq \phi(k_{n+1}) \leq \phi(k_n) + 1$. To choose the next generation of cubes, for each $Q \in E_n$ we compare $f_{k_n}(Q)$ to $\phi(k_{n+1})$:

If $f_{k_n}(Q) < \phi(k_{n+1})$, then choose those N th generation descendants Q' of Q for which $MN \geq f_{k_{n+1}} - f_{k_n} \geq 1$. According to Lemma 5.1 there are at least $\epsilon 2^{dN}$ of them, and then $f_{k_{n+1}}(Q') = f_{k_n} + a$, where $a \in [1, MN]$. Therefore, $|f_{k_{n+1}}(Q') - \phi(k_{n+1})| \leq D$ because

$$(6.1) \quad f_{k_{n+1}} - \phi(k_{n+1}) = f_{k_n} + a - \phi(k_{n+1}) < a \leq MN \leq D$$

and

$$(6.2) \quad \begin{aligned} \phi(k_{n+1}) - f_{k_{n+1}} &= \phi(k_{n+1}) - f_{k_n} - a \\ &\leq \phi(k_n) - f_{k_n} + 1 - a \\ &\leq D + 1 - a \\ &\leq D. \end{aligned}$$

If $f_{k_n}(Q) \geq \phi(k_{n+1})$, then choose those N th generation descendants Q' of Q for which $-MN \leq f_{k_{n+1}} - f_{k_n} \leq -1$. From Lemma 5.1 we know that there are at least $\epsilon 2^{dN}$ of them, and then $f_{k_{n+1}}(Q') = f_{k_n} - a$, where $a \in [1, MN]$. Therefore $|f_{k_{n+1}}(Q') - \phi(k_{n+1})| \leq D$, because

$$(6.3) \quad f_{k_{n+1}} - \phi(k_{n+1}) = f_{k_n} - a - \phi(k_{n+1}) > -a \geq -MN \geq -D$$

and

$$(6.4) \quad \begin{aligned} \phi(k_{n+1}) - f_{k_{n+1}} &= \phi(k_{n+1}) - f_{k_n} + a \\ &\geq \phi(k_n) - f_{k_n} - 1 + a \\ &\geq -D + a - 1 \\ &\geq -D. \end{aligned}$$

So we define the next imbedded set E_{n+1} by

$$(6.5) \quad E_{n+1} = \bigcup_{Q \in E_n} \{\text{all the chosen descendants of } Q\}.$$

Then $E_{n+1} \subset E_n$, and for all $Q' \in E_{n+1}$ we have $|f_{k_{n+1}}(Q') - \phi(k_{n+1})| \leq D$; moreover,

$$(6.6) \quad |E_{n+1} \cap Q|_d \geq \epsilon 2^{dN} \frac{|Q|_d}{2^{dN}} = \epsilon |Q|_d$$

for all $Q \in E_n$. Since $\lim_{t \rightarrow \infty} \phi(t) = \infty$, the inequality $|f_n(Q) - \phi(n)| \leq D$ implies that the quotient $\left| \frac{f_n(Q)}{\phi(n)} \right|$ is also bounded above. Moreover, it is also bounded away from zero for sufficiently large values of n . Therefore, for all $Q \in E_n$,

$$(6.7) \quad \frac{1}{D} \leq \left| \frac{f_{k_n}(Q)}{\phi(k_n)} \right| \leq D.$$

The nested sets $\{E_n\}$ defined this way satisfy the requirements of Lemma 3.3, so the Hausdorff dimension of the set $E = \bigcap_n E_n$ is

$$(6.8) \quad \dim_{\mathbb{H}}(E) \geq d - C(N, \epsilon)$$

with $\lim_{N \rightarrow \infty} C(N, \epsilon) = 0$.

According to Theorem 3 in Sullivan’s paper [17]

$$(6.9) \quad \frac{1}{C} \leq \left| \frac{U(z)}{\text{dist}(z, z_0)} \right| \leq C,$$

and Lemma 4.2 states that $|U(z) - f_{Q_z}| \leq A$, which we can combine with the inequality (6.7), $\frac{1}{D} \leq \left| \frac{f_n(Q)}{\phi(n)} \right| \leq D$, to get

$$(6.10) \quad \frac{1}{C'} \leq \left| \frac{\text{dist}(\gamma(n), z_0)}{\phi(n)} \right| \leq C'.$$

This shows that $\dim_{\mathbb{H}}(\Lambda_\phi) = d$. □

Choosing $\phi(t) = t$ the set Λ_ϕ determines the deep points, and using Theorem 6.1 we get the following corollary:

Corollary 6.3. *If G is a non-compact, geometrically infinite and topologically tame Kleinian group, $M = \mathbb{B}/G$ has injectivity radius bounded away from zero, and there exists a Green’s function on M , then the deep points have dimension 2.*

Similarly in the Fuchsian case:

Corollary 6.4. *If G is a geometrically infinite Fuchsian group so that there exists a Green’s function on $M = \mathbb{B}/G$ and the injectivity radius is bounded away from zero as well as bounded from above, then the deep points have dimension 1.*

The dimension of escaping points has been studied, and recently J. L. Fernández and M. V. Melián (in [9]) showed that the escaping geodesics on a finitely generated divergence type Riemann surface have dimension 1. Deep points form a smaller set, but perhaps one could derive a similar theorem for an intermediate set, for Λ_ϕ as defined in Section 1 with some function ϕ .

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