Research Statement

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1. Overview of past research

My research interests lie at the interface of Algebraic Geometry and Teichmüller dynamics. The moduli space of curves $\mathcal{M}_g$, parametrizing Riemann surfaces of genus g, has been a central object in algebraic geometry for almost a century and studying its geometry is still an active field of research. The moduli space itself is not compact and has a natural compactification, the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$, obtained by adding nodal Riemann surfaces in the boundary. Degenerating smooth Riemann surfaces or equivalently smooth projective curves towards the boundary of $\overline{\mathcal{M}}_g$ has proven to be a crucial tool for studying $\mathcal{M}_g$. Although it seems contradictory at first, even when studying a problem about smooth Riemann surfaces; it is often easier to reduce it to a problem about nodal curves. This allows transforming algebraic problems into problems of combinatorial nature. For example, degenerations to nodal curves can be used to deduce the Gieseker-Petri theorem in Brill-Noether theory [Gie82, EH83] and the celebrated result that $\mathcal{M}_g$ is of general type [HM82].

A key object in Teichmüller dynamics are *flat surfaces*. A flat surface is a Riemann surface together with a holomorphic one-form. As for the moduli space of curves it has proven beneficial to consider the collection of all flat surfaces rather than a specific one. The moduli space of all flat surfaces, with fixed multiplicities of zeros of the 1-form, is called a *stratum* of differentials and denoted by $\mathcal{H}_g$. The stratum $\mathcal{H}_g$ admits a local coordinate system, called *period coordinates*, given by periods of the differential and furthermore a natural $\mathrm{GL}(2,\mathbb{R})$-action.

One of the main goals in Teichmüller dynamics is the classification of orbit closures for the $\mathrm{GL}(2,\mathbb{R})$-action. By recent breakthroughs of Eskin-Mirzakhani [EM13], Eskin-Mirzakhani-Mohammadi [EMM15] and Filip [Fil16] orbit closures are *algebraic* subvarieties of strata that locally in period coordinates are given by linear equations with real coefficients. Orbit closures have several more remarkable properties. For example they are always varieties defined over a number field, raising interesting arithmetic questions and are also related to the geometry of Teichmüller space $\mathcal{T}_g$, the universal cover of $\mathcal{M}_g$. In fact $\mathcal{T}_g$ carries a natural *Teichmüller metric* and its geodesic flow can be related to the $\mathrm{GL}(2,\mathbb{R})$-action. So far the study of orbit closures have been studied extensively from the viewpoint of dynamical systems. The results by Eskin, Mirzakhani, Mohammadi and Filip open the door to approach the classification problem for orbit closures using algebraic geometry.

A main theme in my research is the application of degeneration techniques for differential forms to both algebro-geometric questions about strata as well as problems in Teichmüller dynamics. On one hand, I have studied the behavior of orbit closures for the $\mathrm{GL}(2,\mathbb{R})$-action near the boundary of the moduli space. One of my main results in [Ben20a] is that, roughly speaking, the boundary of an orbit closure "looks like an orbit closure of smaller dimension". More precisely, I showed that the boundary is an algebraic variety of smaller dimension which is locally defined by linear equations with real coefficients. In a subsequent paper [BDG20] with Samuel Grushevsky and Benjamin Dozier we continue the study of the boundary and obtain further restrictions on the possible type of linear equations of orbit closures. I envision applying this to the classification problem of orbit closures in a recursive manner.

My techniques apply not only to orbit closures but also in the more general context of *linear subvarieties* of strata. These are algebraic subvarieties of strata which are locally in period coordinates defined by linear equations with arbitrary complex coefficients. Interesting examples, besides orbit closures, are given by Hurwitz spaces or more generally, loci of rational functions with prescribed ramification at some marked points. The main idea is that a rational function with prescribed ramification is equivalent to an exact differential with prescribed orders of vanishing. Exact differentials can be characterized by the vanishing of all absolute periods; a condition that is given by linear equations in periods coordinates. A particular example I have studied are *double ramification loci*, i.e. loci of rational functions with prescribed ramification over 0 and ∞. In [Ben20b] I describe one such compactification explicitly in geometric terms using exact differentials. In the future I am planning to use the compactification to study the geometry of double ramification loci in more detail.

2. The boundary of linear subvarieties

Near a flat surface $(X,\omega) \in \mathcal{H}_g$ the stratum is locally isomorphic to the relative cohomology $H^1(X, Z(\omega))$ relative to the zeros of ω. The local isomorphism is given by the map $\gamma \mapsto \int_\gamma \omega$ integrating ω over a relative homology class γ. By choosing a relative homology basis we can then identify $H^1(X, Z(\omega))$ with $\mathbb{C}^n$ and in this way obtain the period coordinates. The action of $\mathrm{GL}(2,\mathbb{R})$ can be now described by acting on the real and imaginary part of $\int_\gamma \omega$, i.e. we write $\mathbb{C}^n \cong \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{R}^2$ and $\mathrm{GL}(2,\mathbb{R})$ acts simply on $\mathbb{R}^2$ by matrix multiplication. For almost all surfaces in a stratum $\mathcal{H}_g$ their $\mathrm{GL}(2,\mathbb{R})$-orbit is dense inside the stratum. Nonetheless, in rare circumstances the orbits for the $\mathrm{GL}(2,\mathbb{R})$-action happen to be already closed. In this case they project to algebraic curves in $\mathcal{M}_g$ which are called *Teichmüller curves*. Teichmüller curves are the orbit closures of smallest dimension and they have been studied extensively from both algebro-geometric and dynamical points of view. Although we know by Filips results that orbit closures are algebraic varieties in general, so far the study of higher dimensional orbit closures uses mostly dynamical methods and the use of algebraic geometry has been limited.

One of the reasons is that orbit closures are never compact and many algebro-geometric methods, in particular intersection theory, work best in the context of projective or compact varieties. We are thus led to the problem of constructing a suitable compactification of orbit closures. Once such a compactification has been constructed an inductive framework for studying the classification problem can be setup as follows. We start with an orbit closure M of dimension n and consider its boundary. Part of my results in [Ben20a] is that the boundary ∂M has the structure similar to an orbit closure since it is still locally defined by linear equations and additionally its dimension is one less. If we have already classified all varieties of dimension $n-1$ locally defined by linear equations, this should then in turn allow us to restrict the possibilities for M.

In [MW15, CW19] the authors construct a flat geometric partial compactification of strata, the so-called *"WYSIWYG"-partial compactification* $\hat{\mathcal{H}}_g$ and show that the boundary of an orbit closure in $\hat{\mathcal{H}}_g$ is a finite union of orbit closures. A caveat of $\hat{\mathcal{H}}_g$ is that it is not an algebraic variety by results of [CW19]. Recently a smooth, algebraic, modular compactification $\Xi\overline{\mathcal{M}}_g$ of the stratum $\mathcal{H}_g$ has been constructed in [BCGGM19b], the *moduli space of multi-scale differentials*. A simple way to construct an algebraic compactification of an orbit closure is to take the closure inside the moduli space of multi-scale differentials. My first result is concerned with an explicit description of the boundary of an orbit closure inside the moduli space of multi-scale differentials. This can be seen as a first step in the program of constructing a "nice" compactification of orbit closures.

A boundary point (X,ω) of the moduli space of multi-scale differentials consists of a nodal Riemann surfaces X together with a collection of meromorphic differential forms ω on each irreducible component X_v of X. The dual graph Γ of a nodal curve X is the graph consisting of vertices for each irreducible component and an edge connecting two vertices if the corresponding irreducible components are connected by a node. On $\Xi\overline{\mathcal{M}}_g$ the dual graph Γ of every boundary point (X,ω) additionally has the structure of a level graph, i.e. to every vertex we assign an integer, called the *level* depending on the growth rate of ω along a degenerating family. We call a node *horizontal* if it connects two vertices of the same level and *vertical* otherwise.

A key feature of the moduli space of multi-scale differentials is that its boundary is stratified by level graphs and each boundary stratum behaves like a product of different strata of lower genus. In particular, each boundary stratum has an analog of period coordinates. Thus it is natural to ask: What are the defining equations for the boundary of an orbit closure M? Are they again linear? My first result gives an affirmative answer to this question.

Theorem 2.1 ([Ben20a])**.** *The boundary of M inside $\Xi\overline{\mathcal{M}}_g$, intersected with any open boundary stratum, is an algebraic variety which locally in period coordinates is given by* levelwise *linear equations with real coefficients.*

Levelwise here means that the linear equations only restrain periods contained in components of the same level. The idea is that if two periods are at different levels, along a degenerating family one grows much faster than the other one and so in the limit we only see the period contained in the highest level. More precisely, I obtain an explicit way of obtaining the linear equations of the boundary from the equations on M. We start with a point $(X_0, \omega_0) \in M$ and a boundary point $(X, \omega) \in \partial M$. Each node e of X is obtained by pinching a corresponding homology class in X_0. We call this class the vanishing cycle of e. Sine X is obtained as a quotient of X_0, there is a natural restriction map in homology. Now M is, locally near (X_0, ω_0), defined by linear equations in period coordinates. Sine the stratum is locally isomorphic to $H^1(X_0, Z(\omega_0))$ we can interpret a linear equation for M as a relative homology class. We say that an equation F for M *crosses a node* e if it has non-zero intersection number with the vanishing cycle of e. Now we can describe the linear equations for the boundary ∂M as follows. Given a linear equation for M near (X_0, ω_0), if F crosses some horizontal node then we forget F. Otherwise we restrict F to X and the restriction is a linear equation for ∂M. In that way we obtain all linear equations defining ∂M.

The statement of our results is similar to the boundary description in [CW19] and in fact as a byproduct of my techniques developed in [Ben20a] I am able to reprove the main theorem of [CW19].

My results give a complete description of the local defining equations of the closure $\overline{M}$ intersected with each boundary stratum. To complete the description of $\overline{M}$ it would be desirable to describe local analytic equations for $\overline{M}$ in $\Xi\overline{\mathcal{M}}_g$. Understanding the analytic equations for $\overline{M}$ goes hand in hand with understanding which nodes of a boundary point (X, ω) can be smoothed out while staying inside the boundary of M.

In a subsequent paper [BDG20] with Ben Dozier and Samuel Grushevsky we determine completely which collection of nodes can be smoothed out while staying in the boundary. A particular case of our result is the following.

Theorem 2.2 ([BDG20])**.** *Let M be an orbit closure and (X, ω) a boundary point of M with level graph Γ.*

Locally near the boundary there exists a basis $\mathfrak{B}$ for the linear equations of M such that for each equation $F \in \mathfrak{B}$ the following is true.

(1) *For every pair of horizontal nodes crossed by F the corresponding vanishing cycles are proportional on M.*
(2) *There exists a linear relation relating all vanishing cycles of vertical nodes crossed by F.*

The result should be understood as a way of restricting the possible types of linear equations for an orbit closure. If there exists a boundary point and a linear equation crossing through some collection of nodes, this forces several further linear relations to be satisfied on M.

A different result in the literature imposing restrictions on the type of linear equations is given by Wrights cylinder deformation theorem [Wri15]. Using our results we are able to reprove the cylinder deformation theorem and extend it to the case of strata of meromorphic differentials.

What we actually prove goes beyond the statement of Theorem 2.2. In fact we we completely determine which set of nodes can be smoothed out while staying inside $\overline{M}$. On the way we determine the analytic equations of an orbit closure near the boundary completely. In particular we see that the orbit closure can be singular at the boundary but the singularities are relatively mild.

Towards a classification of orbit closures. The first direction for future work is applying our results on the boundary structure to the classification of orbit closures. In genus 2 orbit closures have been classified by McMullen and Calta [McM03, Cal02], while the classification is still open in genus 3 and higher, see for example [ANg16, NW14, LN14] for partial results. The boundary of an orbit closure will in general not be an orbit closure for the $\mathrm{GL}(2, \mathbb{R})$-action anymore; nonetheless we have seen that it is still locally defined by linear equations. Thus we have now the necessary tools to setup our inductive framework for studying orbit closures.

It was conjectured by Mirzakhani that all higher dimensional orbit closures are trivial in the sense that the holomorphic differential forms are just the pullback under a branched covering with fixed ramification. While the conjecture turned out to be false in general, see recent counterexamples in [MMW20, EMMWW20], there are still indications that non-trivial orbit closures are rare. Thus one possible problem is to find conditions under which an orbit closure is trivial. For example a tangible question is, if the boundary of an orbit closure M is trivial, when is the same true for M?

(Non)-Algebraicity of complex-linear subvarieties. The majority of our results from [Ben20a, BDG20] apply not only in the context of orbit closures for the $\mathrm{GL}(2,\mathbb{R})$-action but work in the general context of algebraic *linear subvarieties*. We call an analytic subvariety of a stratum *linear* if locally in period coordinates it is given by linear equations with arbitrary complex coefficients. By work of Filip [Fil16] every linear subvariety in a stratum of holomorphic differentials where the linear equations have real coefficients is algebraic. The same problem is still open for arbitrary linear equations and strata of meromorphic differentials. It is natural to conjecture the following.

Conjecture 2.3. *An analytic subvariety of a stratum of holomorphic differentials which is locally in period coordinates given by linear equations, is algebraic.*

The proof in [Fil16] inherently use properties of the $\mathrm{GL}(2,\mathbb{R})$-action and thus does not apply to linear subvarieties with complex coefficients. My results do not make use of the $\mathrm{GL}(2,\mathbb{R})$-action or the fact that the equations are defined over the real numbers. I thus believe that my results could shed some light on this conjecture. A first step in this direction would be to try and reprove the results of [Fil16] using our new methods. A potential approach to show algebraicity of linear subvarieties is using results from o-minimality. Linear equations are definable in an o-minimal structure and to show algebraicity it thus suffices to cover a complex linear subvariety by *finitely many* linear subspaces. This problem is potentially related the existence and finiteness of a linear measure supported on complex linear subvarieties.

On the other hand, [BM19] found an example of a linear subvariety in a stratum of meromorphic differentials with linear equations defined over the real numbers, that is not an algebraic variety and it is natural to ask if there are any conditions one can impose on the linear equations to guarantee that a complex linear subvariety is algebraic.

"Nice" compactifications of orbit closures. By taking the closure of an orbit closure or more generally a complex linear subvariety inside the moduli space of multi-scale differentials we have described a suitable compactification where the boundary can be described in geometric terms. Our results from [BDG20] show that in certain cases this compactification is smooth, for example if the limit differentials have no residues. But in general the analytic equations will be singular. Thus our results so far can only be seen as the first step in a larger program of constructing a smooth compactification of orbit closures and I plan to apply the degeneration techniques available to this problem. For Teichmüller curves intersection numbers with the boundary of $\overline{\mathcal{M}}_g$ can be related to various dynamical properties of the $\mathrm{SL}(2,\mathbb{R})$-action. A thorough understanding of a suitable compactification of orbit closures and the corresponding intersection theory would allow to generalize these relations.

3. Degenerations of rational functions and double ramification cycles

For every $k \geq 0$ and a partition $m = (m_1, \dots, m_n)$ of $k(2g-2)$ there exists a natural subvariety

$$\mathcal{H}_g^k(m) := \left\{ (X, p_1, \dots, p_n) \,:\, \mathcal{O}_X\left(\sum_{k=1}^{n} m_i p_i\right) \simeq \omega_X^{\otimes k} \right\} \subseteq \mathcal{M}_{g,n}$$

of the moduli space of curves with marked points. For $k \geq 1$ they are called strata of k-differentials and $k = 1$ is just the case of strata of meromorphic differentials. In the case $k = 0$ the variety $\mathcal{H}_g^0(m)$ is called a *double ramification locus* which consists of marked Riemann surfaces such that there exists a rational function $f : X \to \mathbb{P}^1$ with prescribed order of zeroes and poles at the marked points. Equivalently, $\mathcal{H}_g^0(m)$ can be described as the pullback of the zero section of the universal Jacobian under the Abel-Jacobi map. Eliashberg posed the problem of constructing a meaningful extension of double ramification loci to $\overline{\mathcal{M}}_{g,n}$ for the development of symplectic field theory and the problem has attracted tremendous interest since then. Motivated by Gromov-Witten theory, a possible extension is given by pushing forward the virtual fundamental class of the moduli space of rubber maps to $\mathbb{P}^1$, see [Li01, Li02, GV05]. A different approach is trying to extend the Abel-Jacobi section to $\overline{\mathcal{M}}_{g,n}$, see [HKP18, Hol19, MW17]. All these different extensions yield the same cycle class and a formula for the class in the tautological ring of $\overline{\mathcal{M}}_{g,n}$ was conjectured in 2014 by Pixton and later proved by [JRPZ17]. Yet another approach to extend double ramification cycles is to use the stack of admissable covers. Using the stack of admissable covers one can obtain a description of the closure of double ramification loci inside $\overline{\mathcal{M}}_g$ in terms of the existence of admissable covers but the existence of such is hard to verify in practice.

We thus suggest a different method that gives a more concrete description and also connects double ramification cycles and strata of differentials.

I obtain a geometric description of the boundary of $\mathcal{H}_g^0(m)$ in terms of *twistable rational functions*. A twistable rational function f on a nodal curve X is a collection of rational functions on each component X_v such that the order of vanishing at every marked point is exactly m_k. Additionally there exists a balancing condition for the ramification of the rational function at the nodes. Similarly to the moduli space of multi-scale differentials twistable rational functions also come with a level function on the dual graph Γ of X. In order for a twistable rational function to lie in the closure of $\mathcal{H}_g^0(m)$ there exists an obstruction, given by the vanishing of the *evaluation morphism*. The evaluation morphism of f takes a path γ in Γ and restricts it to the vertices of the highest level. Afterwards it sums up the values of f at certain marked points and nodes that are crossed by the restriction of γ. Our description of the closure is then as follows.

Theorem 3.1 ([Ben20b])**.** *A stable curve $(X, p_1, \ldots, p_n)$ lies in the closure of $\mathcal{H}_g^0(m)$ if and only if there exists a twistable rational function on $(X, p_1, \ldots, p_n)$ such that the evaluation morphism vanishes identically.*

A rational function is equivalent to an exact differential and exact differentials can be characterized by the vanishing of all absolute periods; a condition that is linear in period coordinates. By associating to a rational function its exact differential we can thus realize $\mathcal{H}_g^0(m)$ as a linear subvariety of a stratum and harness our results about the boundary of linear subvarieties. A direct application of the main theorem in [Ben20a] shows that the vanishing of the evaluation morphism is a necessary obstruction and our further results from [BDG20] then show that it is also sufficient.

The geometry of double ramification loci. While cycle classes of double ramification cycles have been studied extensively, not much is known about their geometry. The connected components have been classified in [KZ03, Boi12] and in [Gen15] uses degeneration techniques for meromorphic differentials to study the geometry of strata and obtains basic information about their Kodaira dimensions. Recently, the Euler characteristics of strata have been computed using the moduli space of multi-scale differentials in [CMZ20a]. I propose a similar program for double ramification loci using our degeneration techniques for rational functions. Some basic questions are:

Problem 3.2. *Describe the irreducible components of $\mathcal{H}_g^0(m)$. What are their Kodaira dimensions?*

The realization problem for double ramification loci. The realization problem in tropical geometry asks whether a tropically defined object can be realized as the tropicalization of an algebraic object. The realization problem for canonical divisors has been solved in [MUW17] using the description of the closure of $\mathcal{H}_g^1(m)$ in [BCGGM18] and it is natural to ask the same question for $\mathcal{H}_g^0(m)$. In [UZ19] the authors introduce a natural tropical double ramification locus.

Question 3.3. *Which tropical curves in the tropical double ramification locus are realizable?*

A characterization in terms of modifications of the tropical curves is obtained in [UZ19]. The need for modifications arises from using admissible covers and prestable curves in their proof. Our results for double ramification loci can be phrased solely in terms of the original curves and without referring to a prestable model; I thus hope to give a more direct answer to the realization problem.

A SAGE package for twistable rational functions. Twistable rational functions have an intricate combinatorial structure and the complexity of their dual graphs grows exponentially. To perform computations and computer experiments it would be useful to list all possible dual graphs for twistable rational functions in a given strata. Twistable rational functions are similar to twisted differentials in many ways and in [CMZ20b] the authors constructed a SAGE-package for computations with twisted differentials. I plan to build on their work and extend the package to being able to handle twistable rational functions.

4. Further questions

Teichmüller curves in genus 3 with non-trivial Forni subspace. This project is ongoing joint work with David Aulicino and Chaya Norton. Our goal is contribute to the classification of orbit closures in genus 3. We focus on a special case of the classification problem, Teichmüller curves with non-trivial Forni subspace. We are trying to apply new, analytic techniques constructing differentials along degenrating surfaces which have been developed in [GKN19, HN18]. These techniques allow to explicitly compute the expansion of the period matrix in a neighborhood of a nodal curve. This been applied successfully in [ANo19] to rule out the existence of Shimura-Teichmüller curves in genus 5. We focus on the special case of Teichmüller curves with non-trivial Forni subspace, that have been studied in [Aul18]. In particular there are only 6 different possible dual graphs for a nodal degeneration of a Teichmüller curve with non-trivial Forni subspace. Without giving the precise definition of the Forni subspace, a consequence is that the derivative of the period matrix along the Teichmüller curve has zero determinant. Using the new expansion techniques we can expand the determinant in an explicit way and if we can show that the determinant does not vanish identically, we can rule out the existence of such Teichmüller curves.

Riemann-Hurwitz existence problem. Given a branched covering $f : \Sigma' \to \Sigma$ between two surfaces, we call the tuple consisting of the degree of f, the number of branch points and the ramification index at the preimage of every branch point the *branching profile* of f. A necessary condition for a branching profile to be realized by a branched cover is that it satisfies the condition of the Riemann-Hurwitz theorem, relating the Euler characteristic of Σ, Σ' and the ramification indices. Hurwitz asked the following question.

Question 4.1. *Given two surfaces Σ and Σ' which branching profiles can be realized by a branched covering $f : \Sigma' \to \Sigma$?*

The problem has been open for almost a century and there has been a lot of partial progess, see [PP06] for the state of the art. There have been positive and negative results and the only remaining case is the case of branched covers of $\Sigma = \mathbb{P}^1$. The techniques that have been used are ranging from geometric group theory to dessins d'enfants; I suggest a new approach using degeneration techniques.

For a given branching profile we can consider the subvariety $M \subseteq \mathcal{M}_{g,n}$ consisting of marked curves $(X, p_1, \ldots, p_n)$ such that there exists a branched covering $f : X \to \mathbb{P}^1$ with prescribed ramification at each p_k. Similarly to the double ramification locus, by considering the exact differential df we can realize M as a linear subvariety of a stratum and thus describe the boundary of M in $\overline{\mathcal{M}}_{g,n}$ in geometric terms. For each irreducible component of a boundary point there exists a branched covering of $\mathbb{P}^1$ with prescribed ramification together with some additional combinatorial conditions. This allows an inductive procedure to realize new branching profiles.

Pairs of differentials. Strata describe Riemann surfaces together with a single one-form. In various geometric situations one would like to consider pairs or tuples of differential forms on the same Riemann surface. These loci can be realized as fiber products of strata over $\mathcal{M}_g$. I want to initiate a program studying those in more detail, starting from their deformation theory to their degenerations. I have two main applications in mind: Given a linear system g^r_d of degree d and rank r, there exists a collection of $r + 1$ rational functions on X, and thus understanding $(r+1)$-tuples of rational functions and their degenerations can be applied to Brill-Noether theory. A similar approach would also work to study Weierstrass semigroups.

The second application is to study multiplication maps. In [Mul19] Mullane produced several extremal and rigid rays in cones of effective divisors using strata of k-differentials. By using multiplication maps

$$\mathcal{H}_g^k(m) \times_{\mathcal{M}_{g,n}} \mathcal{H}_g^l(m') \to \mathcal{H}_g^{k+l}(m+m'),$$
$$(X, \omega_1, \omega_2) \mapsto (X, \omega_1\omega_2)$$

we can produce new subvarieties of $\mathcal{M}_g$ and potentially new extremal or rigid cycles. As a first step one would need to understand the deformation theory of such pairs or tuples in order to determine the dimensions of the image of these multiplication maps.

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